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ON THE IMPORTANCE OF THE RELATION $[(A, B), (A, C)] < (A, [(B, C), (C, A), (A, B)]$ BETWEEN THREE ELEMENTS OF A STRUCTURE

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If A, B, and C are elements of an arbitrary structure then we have always

$$[(A, B), (A, C)] \leqq (A, [(B, C), (C, A), (A, B)]). \tag{1}$$

(We will use in this paper the terminology and notation introduced in the paper (II); the marks $<$ and $>$ shall exclude the equality.) If the structure is especially a so-called Dedekind structure then in (1) the equality holds always i.e. the equation

$$[(A, B), (A, C)] = (A, [(B, C), (C, A), (A, B)]) \tag{2}$$

is generally valid. (See for instance (II), p. 413, equation (4).) In this paper we inquire whether conversely every structure in which the equation (2) is generally valid is a Dedekind structure, and how those structures in which (2) is not generally valid are characterized.

THEOREM 1. *If A, B, and C are elements of a structure, and between the cross-cuts (B, C), (C, A), and (A, B) there is at least one relation of inclusion then* (2) *is valid.*

PROOF. If for instance $(B, C) \leqq (C, A)$, then we have obviously $(B, C) \leqq (A, B)$ too. Therefore it is sufficient to consider the two cases $(B, C) \leqq (C, A)$ and $(C, A) \leqq (B, C)$.

1. Let (B, C) be $\leqq (C, A)$. In this case we have

$$[(B, C), (C, A), (A, B)] = [(C, A), (A, B)];$$

from this equation ensues immediately the asserted equation (2).

2. Let (C, A) be $\leqq (B, C)$. In this case we have obviously

$$[(B, C), (C, A), (A, B)] \leqq B$$

whence

$$(A, [(B, C), (C, A), (A, B)]) \leqq (A, B),$$

and consequently

$$(A, [(B, C), (C, A), (A, B)]) \leqq [(A, B), (A, C)].$$

But the last inequality implies together with the inequality (1) the asserted equation (2).

THEOREM 2. *If between the elements A, B, and C of a structure there is at least one relation of inclusion then* (2) *is valid.*

Theorem 2 ensues immediately from Theorem 1.

There are non-Dedekind structures in which nevertheless (2) is satisfied for arbitrary elements A, B, and C. We have for instance the

THEOREM 3: *In any non-Dedekind structure of the fifth order* (2) *is generally valid.*

PROOF. If Σ is a non-Dedekind structure of the fifth order then we can (see (I), §6) denote its elements by P, Q, R, S, and T in such a way that the following inequalities hold:

$$(3) \qquad \begin{cases} P < Q, P < R, P < S, P < T; \\ Q < T; \\ R < S, R < T; \\ S < T. \end{cases}$$

From (3) it is directly evident that between any three elements of Σ there is always at least one relation of inclusion. Now it ensues immediately from Theorem 2 that Theorem 3 is true.

Theorem 3 will be still further generalized in the following.

THEOREM 4. *There are structures in which* (2) *is not generally valid.*

PROOF. We consider a set Λ consisting of 9 elements E, L, M, N, X_1, Y, Z, X, F. Now we will establish between these elements of Λ the following 24 proper relations of inclusion:

$$(4) \qquad \begin{cases} E < L, E < M, E < N, E < X_1, E < Y, E < Z, E < F; \\ L < Y, L < Z, L < F; \\ M < Z, M < X_1, M < X, M < F; \\ N < X_1, N < X, N < Y, N < F; \\ X_1 < X, X_1 < F; \\ Y < F; \\ Z < F; \\ X < F. \end{cases}$$

One persuades himself easily that by the establishments (4) Λ becomes a partly ordered set. (See also figure on p. 575.) Moreover this partly ordered set is even a structure, and cross-cut and union of any two distinct elements of this structure can be found from the table on p. 576.

According to this table we have

$$[(X, Y), (X, Z)] = [N, M] = X_1,$$

and

$$(X, [(Y, Z), (Z, X), (X, Y)]) = (X, [L, M, N]) = (X, [Z, N]) = (X, F) = X.$$

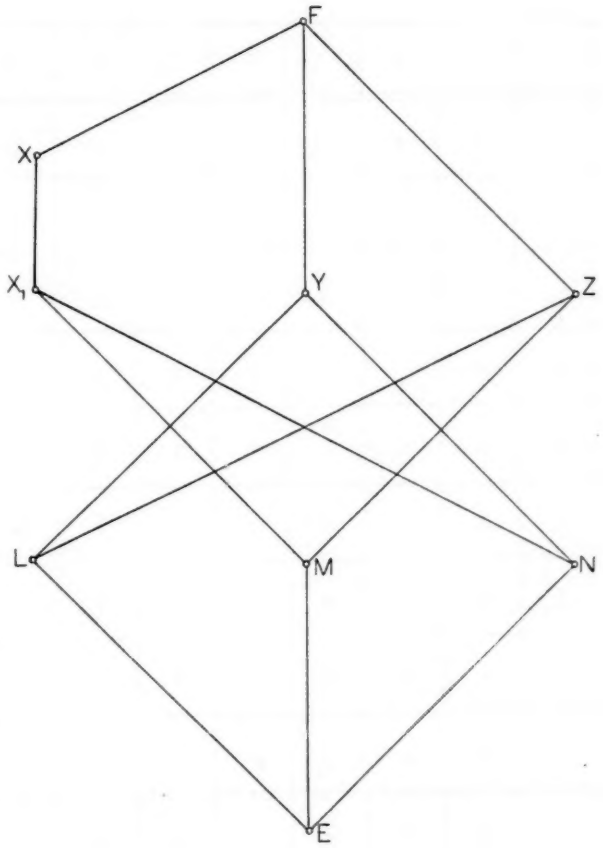
Hence

$$[(X, Y), (X, Z)] \neq (X, [(Y, Z), (Z, X), (X, Y)]).$$

In the structure Λ considered here the equation (2) is consequently not generally valid. (On the other hand one can easily persuade himself that the triples

X, Y, Z and X, Z, Y are the only triples of elements of Λ which do not satisfy the equation (2).)

In the following the marks $E, L, M, N, X_1, Y, Z, X, F$, and Λ shall have always the significance stated.



THEOREM 5. *If A, B, and C are elements of an arbitrary structure Σ then the set Λ' of the 9 elements $E', L', M', N', X_1', Y', Z', X', F'$ defined by the equations*

$$
(5)\qquad \begin{cases} E' = (A, B, C) \\ L' = (B, C) \\ M' = (C, A) \\ N' = (A, B) \\ X_1' = [(C, A), (A, B)] \\ Y' = [(A, B), (B, C)] \\ Z' = [(B, C), (C, A)] \\ X' = (A, [(B, C), (C, A), (A, B)]) \\ F' = [(B, C), (C, A), (A, B)] \end{cases}
$$

is a substructure of Σ, *and if we assign to every element of* Λ *the corresponding accented element of* Λ', *then this correspondence is a homomorphism between* Λ *and* Λ'.

To prove Theorem 5 we have to demonstrate that the table below continues to be valid if we replace all elements appearing in it by the corresponding

	E	L	M	N	X_1	Y	Z	X	F	[ ]
E		L	M	N	X_1	Y	Z	X	F	E
L	E		Z	Y	F	Y	Z	F	F	L
M	E	E		X_1	X_1	F	Z	X	F	M
N	E	E	E		X_1	Y	F	X	F	N
X_1	E	E	M	N		F	F	X	F	X_1
Y	E	L	E	N	N		F	F	F	Y
Z	E	L	M	E	M	L		F	F	Z
X	E	E	M	N	X_1	N	M		F	X
F	E	L	M	N	X_1	Y	Z	X		F
( )	E	L	M	N	X_1	Y	Z	X	F	

accented elements. As this demonstration involves no difficulties we will content ourselves to prove the equation $[X', Y'] = F'$.

We have obviously $X' \geqq (C, A)$. Hence

$$[X', Y'] \geqq [(B, C), (C, A), (A, B)]$$

or

$$[X', Y'] \geqq F'. \tag{6}$$

On the other side we have $X' \leqq F'$ and $Y' \leqq F'$, and consequently also

$$[X', Y'] \leqq F'. \tag{7}$$

(6) and (7) yield the equation

$$[X', Y'] = F'$$

we have wanted to prove.

THEOREM 6. (Lemma.) *Let to every element U of a structure Σ be assigned an element U' of another structure Σ', where this correspondence is a homomorphism between Σ and Σ'. If $U < V$ implies $U' < V'$ then the given homomorphism is an isomorphism.*

PROOF. If $U \neq V$, and between U and V there exists a relation of inclusion then we have, according to the supposition, also $U' \neq V'$. But if we have $U \neq V$ without a relation of inclusion existing between U and V then we have certainly

$$(U, V) < U.$$

Therefore we have, according to what we have supposed in Theorem 6, also

$$(U', V') < U'.$$

The last inequality implies that also in this case $U' \neq V'$. Hence to distinct elements of Σ correspond always distinct elements of Σ'.

THEOREM 7. *If A, B, and C are elements of a structure Σ which do not satisfy the equation* (2) *then the homomorphism between Λ and Λ' existing according to Theorem* 5 *is an isomorphism.*

PROOF. According to Theorem 6 it is sufficient to demonstrate that none of the inequalities (4) turns into an equality if we replace both its sides by the corresponding accented elements. Moreover we can restrict ourselves to prove the 13 inequalities

$$(8)\qquad \begin{cases} E' < L', & E' < M', & E' < N', \\ L' < Y', & L' < Z', \\ M' < Z', & M' < X_1', \\ N' < X_1', & N' < Y', \\ X_1' < X', \\ Y' < F', \\ Z' < F', \\ X' < F', \end{cases}$$

because the other inequalities coming into question, namely the inequalities

$$(9)\qquad \begin{cases} E' < X_1', & E' < Y', & E' < Z', & E' < X', & E' < F', \\ L' < F', \\ M' < X', & M' < F', \\ N' < X', & N' < F', \\ X_1' < F' \end{cases}$$

ensue from the inequalities (8). The discussion shall be divided into 4 steps.

1. If the equation $E' = L'$ were valid we should have

$$(A, B, C) = (B, C)$$

and consequently

$$(B, C) \leqq (A, B);$$

hence, according to Theorem 1, the elements A, B, and C would satisfy the equation (2), contrarily to the supposition of Theorem 7. In corresponding way we can refute the hypotheses $E' = M'$ and $E' = N'$.

2. $L' = Y'$ or $(B, C) = [(A, B), (B, C)]$ would imply $(A, B) \leqq (B, C)$. Again it follows from Theorem 1 that the hypothesis is absurd. Similarly also the hypotheses $L' = Z'$, $M' = Z'$, $M' = X_1'$, $N' = X_1'$, and $N' = Y'$ yield contradictions.

3. According to the supposition of Theorem 7 we have $X_1' \neq X'$.

4. If the equation $X' = F'$ were valid then we should have

$$(L', X') = (L', F'),$$

and consequently

$$E' = L'.$$

This would contradict what we have proved under 1. Just so from the hypothesis $Y' = F'$ we can derive the equation $E' = M'$, from the hypothesis $Z' = F'$ the equation $E' = N'$, and by this a contradiction to what we have proved under 1.

Now the following Theorems 8, 9, and 10 are evident.

THEOREM 8. *Any structure in which the equation* (2) *is not generally valid contains at least one sub-structure of the ninth order having the same property.*

THEOREM 9. *Any structure of the ninth order in which the equation* (2) *is not generally valid is isomorphic with the structure* Λ.

THEOREM 10. *In any structure of at most eighth order the equation* (2) *holds for arbitrary elements* A, B, *and* C.

Theorem 10 represents the announced generalization of Theorem 3.

It is evident that we could investigate the equation

$$([A, B], [A, C]) = [A, ([B, C], [C, A], [A, B])] \tag{10}$$

dually corresponding to the equation (2) exactly in the same way as we have just explored the equation (2). Yet, while the Dedekind axiom

$$[(A, [B, C]), (B, C)] = ([A, (B, C)], [B, C])$$

is identical with its dual counterpart the corresponding assertion about the equation (2) is not true.

THEOREM 11. *In our structure* Λ *the equation* (10) *is generally valid.*

The proof follows easily from the dual counterpart of Theorem 9. Of course Theorem 11 can also be verified simply by the table on p. 576.

THEOREM 12. *The assertion that in a structure the equation* (2) *holds for arbitrary*

elements, A, B, and C, and the assertion dually corresponding to this assertion are not equivalent.

PROOF. See Theorem 11.

PRAGUE

References

(I) Richard Dedekind, *Über die von drei Moduln erzeugte Dualgruppe*. Mathematische Annalen 53(1900), p. 371–403.

(II) Oystein Ore, *On the foundation of abstract algebra*. I. Annals of Mathematics 36(1935), p. 406–437.

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ON THE THEORY OF PARTIALLY ORDERED LINEAR SYSTEMS AND LINEAR SPACES

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In the present work we consider some problems concerning the theory of partially ordered linear systems and linear spaces, which have been developed by F. Riesz, H. Freudenthal, L. Kantorovitch, M. Krein, Sh. Kakutani and others. The main purpose of this paper is to reveal the rôle of a group of axioms analogous to the so called Axiom of Archimedes (or Eudoxe) for real numbers. The strongest of these axioms—Axiom A—determines a class of partially ordered linear systems which proves to be rather important in various aspects of the theory.

§1 contains the complete list of notions and axioms which we shall deal with in the following. As to the elementary properties of the linear system connected with the ordering[1] we refer chiefly to H. Freudenthal [1] and L. Kantorovitch [2].

§1. Notations and definitions

Let $E = \{x, y, z, \cdots\}$ be a *linear system* (an Abelian group with real numbers as operators [3]) and a *partially ordered set*, i.e. between some pairs of elements x, y, of E ($x \neq y$) a relation $x < y$ is defined which satisfies the usual conditions:

1. $x < y$ and $y < x$ are inconsistent;
2. $x < y$ and $y < z$ imply $x < z$.

We shall use the following notations:

$x > y$ denotes that $y < x$.

$x \leqq y$ (or $y \geqq x$) denotes that either $x < y$ or $x = y$.

$x \,||\, y$ denotes that $x \neq y$ and neither $x < y$ nor $x > y$ takes place; we say that x and y are *incomparable*.

$x \in M$ denotes that x is an element of the set M.

$x \bar{\in} M$ denotes that the set M does not contain x.

$M_1 \subset M_2$ (or $M_2 \supset M_1$) denotes that M_1 is a subset of M_2 (M_1 and M_2 may coincide).

$M_1 \leqq M_2$ (or $M_2 \geqq M_1$) denotes that $x \leqq y$, whenever $x \in M_1$, $y \in M_2$.

$N(M)$ denotes the set of all $x \leqq \mathfrak{M}$.

$H(M)$ denotes the set of all $y \geqq \mathfrak{M}$.

We introduce the notation $E^+(E^-)$ for the set $H(O)$ (resp. $N(O)$), where O is the null-element of the linear system E. Every element of E^+ (resp. E^-) except O is called *positive* (resp. *negative*).

Every element $x \in H(M)$ is called an *upper bound*, abbreviated: u.b., of the set M. Every $x \in N(M)$ is called a *lower bound*, abbreviated: l.b., of $\mathfrak{M}$.

The set M is *bounded* if there exist an u.b. as well as a l.b. of M.

[1] We always say "ordering" instead of "partial ordering."

The element x is the *least upper bound*, abbreviated: l.u.b., of the set M if $x \in H(M)$ and $x \leqq H(M)$.

The element $x \in N(M)$ such that $x \geqq N(M)$ is called the *greatest lower bound*, abbreviated: g.l.b., of M.

The l.u.b. (g.l.b.) of the set M we denote by sup M (resp. inf. M). If M contains only a finite number of elements $x_1, x_2, \cdots, x_r$, then we use the notations

$$\sup M = x_1 \vee x_2 \vee \cdots \vee x_r .$$

$$\inf M = x_1 \wedge x_2 \wedge \cdots \wedge x_r .$$

$M_1 \cup M_2$ ($M_1 \cap M_2$) denotes the union (resp. the intersection) of the sets M_1 and M_2.

$\bigcup_\alpha M_\alpha$ ($\bigcap_\alpha M_\alpha$) denotes the union (resp. the intersection) of the system of sets $\{M_\alpha\}$.

$M_1 \dot{-} M_2$ denotes the set of such x that $x \in M_1$, $x \bar{\in} M_2$.

$M_1 \pm M_2$, where M_1, M_2 are subsets of a linear system, denotes the set of the elements $x \pm y$, where x and y range through M_1 and M_2 respectively.

λM, where M is a subset of a linear system and λ is a real number, denotes the set of elements λx, where $x \in M$.

If x is an element of a linear system then D_x denotes the set of elements λx, where $-\infty < \lambda < +\infty$.

The subset of D_x consisting of λx with $0 < \lambda < +\infty$ is denoted by R_x.

If M is an arbitrary subset of a linear system E then $L(M)$ denotes the *linear hull* of M, i.e. the minimal linear subsystem $E' \subset E$ which contains M.

Return to our system E. The following axioms are always supposed to be satisfied:

AXIOM I. *$x < y$ implies $x + z < y + z$ for every $z \in E$.*

AXIOM II. *If $x < y$ then for every number $\lambda > 0$: $\lambda x < \lambda y$.*

AXIOM III. *For any two elements $x \in E$, $y \in E$ there exists a $z \in E$ such that $z \geqq x$, $z \geqq y$.*

The system E which satisfies Axioms I, II and III is called a *partially ordered linear system.*

Let E' be a linear subsystem of E. E' itself necessarily satisfies Axioms I and II (it is possible, of course, that every two elements of E' are incomparable). As to Axiom III it may be fulfilled or not. We say that E' is a *proper subsystem* of E if E' satisfies Axiom III.

Furthermore we shall use some of the following axioms:

AXIOM H*. *For every $x > 0$ the set Rx has no u.b.'s.*

AXIOM H. *For every $x \in E$ the set D_x is not bounded.*

AXIOM R. *If for a given $x \parallel 0$ the set R_x has an u.b. then there exists also a l.b. of R_x.*

We say, for instance, that the system E is E_H ($E_{H,R}$) if E satisfies Axiom H (resp. Axioms H and R).

It is easy to see that every E_H is E_{H^*}. The system E_{H^*} that is not E_H we call *weakly homogeneous*. The system E_H we call *homogeneous*. We say that E is a *regular* system if E is E_R. If E is weakly homogeneous and regular then we call it *weakly Archimedean*. If E is $E_{H,R}$ then we call it *Archimedean*.

It is easy to see that the weakly Archimedean and Archimedean systems can be characterized respectively by the following axioms:

AXIOM A*. *If for a given $x \in E$ the set R_x has an u. b. then necessarily $x \leqq 0$ or $x \parallel 0$; in the latter case R_x has also a l.b.*

AXIOM A. *If for a given $x \in E$ the set R_x has an u. b. then necessarily $x \leqq 0$.*

We introduce also

AXIOM B*. *E contains such an element $u > 0$ that every $x \in E$ can be represented in the form $x = \lambda u - x'$, where $\lambda \geqq 0$, $x' \in E^+$.*

AXIOM B. *E is a Banach space the unit sphere of which has an u.b.*

AXIOM C*. *Every finite subset M of E has a l.u.b. and a g.l.b.*

AXIOM C. *Every subset M of E that has an u.b. has a l.u.b.*

The system E_{B^*} is said to have an *axial element* u. The system E_{C^*} is a *lattice* according to the terminology of G. Birkhoff [4]. We call E *closed* if E is E_C.[2]

REMARK 1.1. There is an essential difference between Axioms H^*, H, R, A^*, A and Axioms B^*, B, C^*, C. The axioms of the first group are *cogradient*, i.e. every proper subsystem $E' \subset E$ is, for instance, E'_H if E itself is E_H. The axioms of the second group do not possess this property.

REMARK 1.2. In Axioms H^*, R, A^* and A we can replace R_x by its subset $\{nx\}$, where $n = 1, 2, \cdots$. In fact, if

$$x_1 \leqq y,$$

$$x_2 \leqq y$$

and α, β are non-negative numbers with $\alpha + \beta = 1$ then

$$\alpha x_1 \leqq \alpha y$$

$$\beta x_2 \leqq \beta y,$$

whence

$$\alpha x_1 + \beta x_2 \leqq \alpha y + \beta y = (\alpha + \beta)y = y,$$

i.e., every $N(y)$ (as well as every $H(y)$) is convex. Suppose now that

$$nx \leqq y$$

for $n = 1, 2, \cdots$. By Axiom III y can be chosen in E^+, i.e., the inequality holds for $n = 0$ too. If $\lambda \geqq 0$ is arbitrary, then $\lambda = n + \alpha$, where n is a non-negative integer and $0 \leqq \alpha < 1$, and we have $\lambda x = (n + \alpha)x = [(1 - \alpha)n + \alpha(n + 1)]x = (1 - \alpha)nx + \alpha(n + 1)x$. Since

$$nx \leqq y,$$

$$(n + 1)x \leqq y,$$

[2] Such systems are usually called *complete*.

we obtain

$$\lambda x = (1 - \alpha)nx + \alpha(n + 1)x \leqq y.$$

§2. Positive linear functions

The real function $l(x)$ defined on E is called *linear* if for arbitrary $x_1 \in E$, $x_1 \in E$, $x_2 \in E$ and numbers λ_1, λ_2

$$l(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 l(x_1) + \lambda_2 l(x_2).$$

THEOREM 2.1. *Let E' be a linear subsystem of E, l' a linear function defined on E'. There exists a linear function l on E such that $l(x) = l'(x)$, whenever $x \in E'$.*

THEOREM 2.2. *Let E' be a linear subsystem of E. If $E' \neq E$ then there exists a linear function l such that $l \neq 0$ and $l(x) = 0$, whenever $x \in E'$.*

THEOREM 2.3. *For every $x_0 \in E$ there exists a linear function $l(x)$ on E such that $l(x_0) = 1$.*

These theorems are the analogues of the well known theorems in the theory of linear functionals in Banach spaces and can be proved in essentially the same way.

The following lemmas will be useful.

LEMMA 2.1. *Axiom III is equivalent (Axioms I and II being fulfilled) to each of the following conditions:*

(a) $E = L(E^+)$;

(b) *Every $x \in E$ can be represented $x = x' - x''$, where $x' \in E^+$, $x'' \in E^+$.*

(c) *If a linear function $l(x)$ vanishes identically on E^+ then $l = 0$.*

PROOF. Axiom III is evidently equivalent to (b). The equivalence of it to the other conditions can be proved according to the scheme

$$(b) \rightarrow (a) \rightarrow (c) \rightarrow (b).$$

The first and the second steps are also obvious. Suppose (b) is not fulfilled. Then the linear subsystem E' consisting of the elements $x' - x''$, where $x' \in E^+$, $x'' \in E^+$, is a proper subsystem of E and does not coincide with E. Therefore there exists a linear function $l \neq 0$ such that $l(x) = 0$ for all $x \in E'$.

LEMMA 2.2. *If E is a finite-dimensional system then Axiom III is equivalent to the following condition:*

(d) *E^+ contains an inner point of E in the sense of the natural topology in E.*[3]

PROOF. Obvious, since in this case (*d*) is equivalent to (*a*) in Lemma 2.1.

The linear function $p(x)$ on E is called *positive* if $p \neq 0$ and $p(x) \geqq 0$, whenever $x \in E^+$.

THEOREM 2.4. *Let E be a weakly homogeneous system, E' an arbitrary proper*

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[3] By the natural topology of the n-dimensional linear system E we understand the unique topology in E with respect to which the linear operations in E are continuous. It can be introduced by the norm: if $x = \sum_{i=1}^{n} \xi_i x_i$ then $\| x \| = \sum_{i=1}^{n} |\xi_i|$, where $\{x_1, x_2, \cdots, x_n\}$ is the basis of E.

subsystem of E, p' a positive linear function on E'. There exists then a positive linear function p on E such that $p(x) = p'(x)$ for every $x \in E'$.

PROOF. Consider the totality $\sum$ of all complex-valued functions $\sigma(x)$(x in E), with $|\sigma(x)| \equiv 1$. We introduce a *weak topology* in Σ defining the neighbourhoods in Σ as follows: a number $\epsilon < 0$ and a finite aggregate of elements $x_i \in E$ ($i = 1, 2, \cdots, r$) are fixed and the neighbourhood $U(\sigma_0 ; x_1, \cdots, x_r ; \epsilon)$ of $\sigma_0(x)$ is defined as the set of such $\sigma(x)$ that

$$|\sigma(x_i) - \sigma_0(x_i)| < \epsilon \qquad (i = 1, 2, \cdots, r).$$

Let K denote the unit circle in the complex plain. Then Σ is the topological product (in the sense of A. Tychonoff) of the spaces K_x, where x runs over E, and every K_x is K,

$$\Sigma = \prod_x^E K_x, \qquad K_x \equiv K.$$

Since K is bicompact, Σ is also bicompact [5].

Σ contains the subspace Φ consisting of the functions $\varphi(x) = e^{il(x)}$, where $l(x)$ is an arbitrary linear function on E. It can be easily verified that Φ is closed in Σ, consequently Φ is bicompact itself.

Let x_0 be an arbitrary positive element of E. If $x_0 \bar{\in} E'$ then we denote by Π_{x_0} the set of the functions $\varphi(x) = e^{il(x)}$ with $l(x_0) \geqq 0$. If $x_0 \in E'$ then we denote by Π_{x_0} the set of such $\varphi(x) = e^{il(x)}$ that $l(x_0) = p'(x_0)$. It follows from Theorems 2.1 and 2.3 that $\Pi_x \neq 0$ for every $x > 0$. It is also easy to see that each Π_x is closed in Φ.

Consider an arbitrary finite system of the sets Π_x

$$\Pi_{x_1}, \Pi_{x_2}, \cdots, \Pi_{x_n} \qquad (x_i > 0; i = 1, 2, \cdots, n).$$

If E' does not contain any of the elements $x_1, x_2, \cdots, x)$, then the function $\varphi(x) \equiv 1$ belongs to every Π_{x_i} ($i = 1, 2, \cdots, n$). If some of x_i belong to E' then we may suppose that $\{x_1, x_2, \cdots, x_k\} \subset E'$ ($k \leqq n$). We take the finite-dimensional subsystems $E_1 = L(x_1, x_2, \cdots, x_n)$ and $E_2 = L(x_1, x_2, \cdots, x_k)$ of E.

There exists a positive linear function $p_1(x)$ on E_1 that coincides with $p'(x)$ on E_2. Assume the contrary: there would exist then an $x_0 \in E_1$, $x_0 > 0$, such that 1) necessarily $p_1(x_0) > 0$, 2) x_0 is the limit (in the sense of the natural topology in E_1) of a sequence of elements $y_m < 0$ ($y_m \in E_1$, $m = 1, 2, \cdots$). Since E_1 is a proper subsystem of E, the set $E_1^+ = E^+ \cap E_1$ contains an inner point y_0 of E_1 (by Lemma 2.2). Then $-y_0$ is an inner point of $E_1^- = E^- \cap E_1$.

Since E_1^- is evidently convex and x_0 belongs to the boundary of E_1^-, we have

$$\alpha(-y_0) + (1 - \alpha)x_0 \leqq 0,$$

whenever $0 < \alpha \leqq 1$, whence

$$\frac{1 - \alpha}{\alpha} x_0 \leqq y_0.$$

$(1 - \alpha)/\alpha$ takes on all positive values when α ranges through the semi-interval $(0, 1]$. We see that y_0 is an u.b. of Rx_0, x_0 being positive, contrary to the hypothesis that E is E_{H^*}. The existence of p_1 is proved. If $l(x)$ is an arbitrary linear function on E that coincides with p_1 on E_1 then $\varphi(x) = e^{il(x)}$ belongs to every $\Pi_{x_i}(i = 1, 2, \cdots, n)$.

We have proved that the intersection of an arbitrary finite system of closed sets Π_x $(x > 0)$ is not empty. Since Φ is bicompact, there exists $\varphi(x) = e^{ip(x)}$ that belongs to every Π_x $(x > 0)$.

It follows immediately from the definition of Π_x that $p(x)$ is a positive linear function coinciding with $p'(x)$ on $E'^+ = E^+ \cap E'$. Since E' is a proper subsystem of E, i.e. $E' = L(E'^+)$ (by Lemma 2.1), we obtain

$$p(x) = p'(x)$$

everywhere on E'. Theorem 2.4 is proved.

Applying Theorem 2.4 to the case $E' = D_{x_0}$, $p'(\lambda x_0) = \lambda$, where x_0 is an arbitrary positive element of a weakly homogeneous system E, we obtain

THEOREM 2.5. *If E is E_{H^*} then for every $x_0 > 0$ there exists a positive linear function $p(x)$* such that $p(x_0) = 1$.

Let z be an arbitrary element of E incomparable with 0. According to Lemma 2.1, z can be represented in the form

$$z = x - y,$$

where $x > 0$, $y > 0$.[4] Consider the elements

$$z'(\alpha) = -\alpha x + y,$$

$$z''(\beta) = x - \beta y$$

and denote by σ (resp. τ) the least upper bound of the α (resp. β) such that $z'(\alpha) \geqq 0$ (resp. $z''(\beta) \geqq 0$). Since

$$z'(0) = y > 0, \qquad z''(0) = x > 0,$$

$$z'(1) = -x + y = -z \parallel 0, \qquad z''(1) = x - y = z \parallel 0,$$

we obtain the inequalities

$$0 \leqq \sigma \leqq 1,$$

$$0 \leqq \tau \leqq 1.$$

LEMMA 2.3. *Let E be E_{H^*} and $z \in E$ incomparable with 0. Then either*

$$\sigma = 1 \qquad (\tau = 1)$$

for every representation of z as the difference of two positive elements or

$$\sigma < 1 \ (\text{resp. } \tau < 1)$$

for every representation of z.

[4] This representation is evidently not unique.

In the first case, R_z has an u.b. (resp. a l.b.) and for each positive linear function $p(x)$

$$p(z) \leqq 0 \text{ (resp. } p(z) \geqq 0).$$

In the second case, R_z has no u.b.'s (resp. l.b.'s) and there exists a positive linear function $p_0(x)$ such that

$$p_0(z) > 0 \text{ (resp. } p_0(z) < 0).$$

PROOF. 1. Suppose that

$$z = x - y \qquad (x > 0, y > 0)$$

and the corresponding $\sigma = 1$. It follows from the definition of σ that for every $\epsilon > 0$

$$-(1 - \epsilon)x + y \geqq 0,$$

whence

$$\epsilon x - x + y \geqq 0,$$

$$z = x - y \leqq \epsilon x,$$

$$\frac{1}{\epsilon} z \leqq x.$$

Since $\epsilon > 0$ was arbitrary, we obtain

$$R_z \leqq x.$$

Quite analogously $\tau = 1$ implies $R_z \geqq -y$.

2. If $R_z \leqq x_0$ ($R_z \geqq y_0$) and $p(x)$ is an arbitrary positive linear function then

$$p(\lambda z) = \lambda p(z) \leqq p(x_0) \qquad \text{(resp. } \geqq p(y_0))$$

for every $\lambda > 0$, whence

$$p(z) \leqq 0 \qquad \text{(resp. } p(z) \geqq 0).$$

3. Suppose that

$$z = x - y \qquad (x > 0, y > 0),$$

σ and τ being arbitrary. Take $E' = L(x, y)$ and define a linear function p' on E', putting

$$p'(x) = 1, \qquad p'(y) = \eta \geqq 0,$$

where η is not yet determined. It is easy to see that $p'(u)$ thus defined is a positive linear function on E' if and only if η satisfies the inequalities

$$\sigma \leqq \eta \leqq \frac{1}{\tau}. \qquad (\circ)$$

We have, indeed, for every $\epsilon > 0$

$$z'(\sigma - \epsilon) = -(\sigma - \epsilon)x + y \geqq 0,$$

$$z''(\tau - \epsilon) = x - (\tau - \epsilon)y \geqq 0.$$

If p' is non-negative on $E'^{+} = E^{+} \cap E'$ then

$$p'(z'(\sigma - \epsilon)) = -\sigma + \epsilon + \eta \geqq 0,$$

$$p'(z''(\tau - \epsilon)) = 1 - (\tau - \epsilon)\eta \geqq 0.$$

Putting $\epsilon \to 0$, we obtain

$$-\sigma + \eta \geqq 0,$$

$$1 - \tau\eta \geqq 0,$$

or

$$\sigma \leqq \eta \leqq \frac{1}{\tau}.$$

The necessity of $(\circ)$ is proved. It is not more difficult to prove the sufficiency of it. $(\circ)$ includes also the case $\tau = 0$ if we put $1/0 = +\infty$.

Suppose that there exists a representation

$$z = x - y \qquad (x > 0, y > 0)$$

for which $\sigma < 1$. Then it is possible to choose η satisfying $(\circ)$ and such that $\sigma \leqq \eta < 1$. By Theorem 2.4 p' can be extended to a positive linear function p on E. We have

$$p(z) = p'(z) = p'(x - y) = 1 - \eta > 0.$$

If for a certain representation of z

$$z = x - y \qquad (x > 0, y > 0)$$

$\tau < 1$, then $1/\tau > 1$ and we can choose $\eta > 1$ satisfying the inequalities $(\circ)$. Then

$$p(z) = p'(z) = p'(x - y) = 1 - \eta < 0,$$

where p is positive linear function on E that coincides with p' on E'

Combining the results 1., 2. and 3., we obtain the statements of our Lemma.

§3. Archimedean systems and systems of real functions

Let E be E_A. The following lemmas state some characteristic properties of Archimedean systems.

LEMMA 3.1. *E is E_A if and only if for every $x > 0$ the null-element 0 is the g.l.b. of the set R_x.*

PROOF. Suppose that E is E_A. For every $x > 0$

$$0 < R_x.$$

It suffices to show that

$$y \leqq R_x$$

implies

$$y \leqq 0.$$

If

$$y \leqq \lambda x$$

for arbitrary $\lambda > 0$ then

$$\mu y \leqq x,$$

where $\mu = 1/\lambda$ can take on all positive values. According to Axiom A

$$y \leqq 0.$$

Conversely, assume that $0 = \inf R_y$, whenever $y > 0$. If for a certain $x \in E$ there exists an u.b. of R_x then we may suppose that y is positive.

$$R_x \leqq y$$

implies obviously

$$x \leqq R_y$$

and by our hypothesis

$$x \leqq 0,$$

i.e. Axiom A is fulfilled.

REMARK 3.1. It is almost evident that generally $0 = \inf_\alpha \{\lambda_\alpha x\}$, where $x > 0$, $\lambda_\alpha > 0$ and $\inf_\alpha \{\lambda_\alpha\} = 0$.

LEMMA 3.2. *E is E_A if and only if $\lambda_n x \leqq y$, $\lambda_n \to \lambda$, imply $\lambda x \leqq y$.*

PROOF. The sufficiency of the condition is almost evident. If

$$R_x \leqq y$$

then

$$R_y \geqq x$$

and

$$R_{-y} \leqq -x.$$

For $\lambda_n = 1/n$ we have then

$$\frac{1}{n}(-y) \leqq -x.$$

Taking $n \to \infty$ we obtain then

$$0 \leqq -x$$

or

$$x \leqq 0,$$

i.e. Axiom A is fulfilled.

Conversely, suppose that E is E_A and consider elements $x \in E$, $y \in E$ and a sequence $\{\lambda_n\}$ converging to λ such that

$$\lambda_n x \leqq y \qquad (n = 1, 2, \cdots).$$

Without loss of generality we can assume that $\{\lambda_n\}$ is a non-decreasing sequence, i.e.

$$\lambda_n = \lambda - \epsilon_n,$$

where $\epsilon_1 \geqq \epsilon_2 \geqq \cdots$ and $\epsilon_n \to 0 (n \to \infty)$. If this is not the case, then we take a non-decreasing subsequence $\{\lambda_{n_k}\} \subset \{\lambda_n\}$ instead of $\{\lambda_n\}$; if $\{\lambda_n\}$ does not contain any such subsequences then we represent $\lambda_n x$ as $(-\lambda_n)(-x)$ and consider the sequence $\{-\lambda_n\}$.

We have

$$(\lambda - \epsilon_n)x \leqq y,$$

$$\lambda x - y \leqq \epsilon_n x.$$

Since $\epsilon_n \to 0 (n \to \infty)$,

$$0 = \inf \{\epsilon_n x; n = 1, 2, \cdots\}$$

and consequently

$$\lambda x - y \leqq 0,$$

$$\lambda x \leqq y,$$

as was to be proved.

LEMMA 3.3. *If E is an Archimedean system, then for every $z \parallel 0$ and every representation of z as the difference of two positive elements*

$$z = x - y \qquad (x > 0, y > 0)$$

there is always $\sigma < 1$ and $\tau < 1$.

PROOF. Suppose that there exists an element $z \in E$ incomparable with 0 that can be so represented

$$z = x - y \qquad (x > 0, y > 0)$$

that, for instance, $\sigma = 1$. It follows from the definition of σ that for any $\alpha_n < 1$

$$-\alpha_n x + y \geqq 0 \qquad (n = 1, 2, \cdots).$$

Putting $\alpha_n \to 1 (n \to \infty)$ and applying Lemma 3.2, we obtain

$$-x + y \geqq 0$$

which is impossible, because $-x + y = -z \parallel 0$.

As to the inequality

$$\tau < 1$$

it can be proved in the same way.

Let $P = \{p\}$ be an arbitrary set, F an arbitrary linear system of real functions $f(p)$ defined on P. We define the *natural ordering* of F: $f_1 < f_2$ if $f_1(p) \leqq f_2(p)$ for all $p \in P$ and $f_1(p_0) < f_2(p_0)$ for at least one $p_0 \in P$. We assume that for every two functions $f_1 \in F, f_2 \in F$ there exists a function $f_3 \in F$ such that $f_3 \geqq f_1$, $f_3 \geqq f_2$.

THEOREM 3.1. *F is an Archimedean system. Conversely, every E_A is isomorphic to a system of real functions defined on some set P with the natural ordering.*

We say that the partially ordered linear system $E = \{x\}$ and $E_1 = \{y\}$ are *isomorphic* if there exists a one-to-one correspondence $y = \varphi(x)$ between the elements of E and E_1 such that $\varphi(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \varphi(x_1) + \lambda_2 \varphi(x_2)$ and $x_1 < x_2$ implies $\varphi(x_1) < \varphi(x_2)$ and vice versa.

PROOF OF THEOREM 3.1. The first statement is almost trivial. Suppose that there exist $f_1 \in F$, $f_2 \in F$ such that

$$\lambda f_1 \leqq f_2$$

for all $\lambda > 0$. We then have

$$\lambda f_1(p) \leqq f_2(p)$$

for every $p \in P$. Since $\lambda > 0$ is arbitrary, this inequality implies

$$f_1(p) \leqq 0,$$

i.e. $f_1 \leqq 0$ in the sense of the natural ordering of F.

Let E be an arbitrary Archimedean system. Denote by P the totality of all positive linear functions on E. To every element $x \in E$ we let correspond the function $f_x(p) = p(x)$ defined on P.

The correspondence

$$x \to f_x(p)$$

is a homomorphism,

$$\alpha x + \beta y \to f_{\alpha x + \beta y}(p) = p(\alpha x + \beta y) = \alpha p(x) + \beta p(y) = \alpha f_x(p) + \beta f_y(p).$$

Furthermore it is biunivoque. It is sufficient to prove that $f_x \neq 0$, whenever $x \neq 0$, i.e. for every $x \neq 0$ there exists a positive linear function p_0 such that $f_x(p_0) = p_0(x) \neq 0$. For $x > 0$ or $x < 0$ it follows from Theorem 2.5. For $x \parallel 0$ it can be deduced from Lemmas 2.3 and 3.3.

Let $x > 0$. Then for every $p \in P$ $f_x(p) = p(x) \geqq 0$ and by Theorem 2.5 there exists a $p_0 \in P$ such that $f_x(p_0) = p_0(x) > 0$, i.e. $f_x > 0$.

We must show that for every $x \bar{\in} E^+$ $f_x \not> 0$, i.e. there exists such $p_1 \in P$ that $f_x(p_1) = p_1(x) < 0$. It follows from Theorem 2.5 in the case $x < 0$ and from Lemmas 2.3 and 3.3 if $x \parallel 0$.

Thus we see that $x > 0$ if and only if f_x is positive in the sense of natural ordering of $F = \{f_x\}$.

§4. The structure of weakly Archimedean systems

LEMMA 4.1. *If the weakly homogeneous system E is regular (i.e. is E_{A^*}) then for every $z \parallel 0$ and every representation*

$$z = x - y \qquad (x > 0, y > 0)$$

either σ and τ are equal to 1, or $\sigma < 1$ and $\tau < 1$ simultaneously.

PROOF. Assume $z \parallel 0$ and

$$z = x - y \qquad (x > 0, y > 0).$$

Suppose that $\sigma = 1$. According to Lemma 2.3 R_z has an u.b. Since E is regular, there exists also a l.b. of R_z. Applying the same Lemma 2.3, we conclude that $\tau = 1$.

$\tau = 1$ implies $\sigma = 1$ analogously. Lemma 4.1 is proved.

Let E be E_{A^*} and p an arbitrary positive linear function on E. Denote by W_p the set of $z \in E$ for which

$$p(z) = 0$$

and by W—the intersection of all W_p,

$$W = \bigcap_p^P W_p.$$

W is evidently a linear subsystem of E and every element of W is incomparable with 0.

It follows from Lemmas 2.3 and 4.1 that W consists of the elements $z \in E$, $z \parallel 0$, for which D_z is bounded and for every representation

$$z = x - y \qquad (x > 0, y > 0)$$

$\sigma = \tau = 1$.

As to the elements $z \parallel 0$ that do not belong to W, for every such z and each representation

$$z = x - y \qquad (x > 0, y > 0)$$

$\sigma < 1$ and $\tau < 1$. By Lemma 2.3 for a given $z \parallel 0$, $z \bar{\in} W$, there exist such $p_1 \in P$, $p_2 \in P$ that $p_1(z) > 0$, $p_2(z) < 0$.

LEMMA 4.2. *If E is E_{A^*,C^*} then E is an Archimedean system.*

PROOF. Suppose that E satisfies Axiom A^* and is a lattice. If x and y are elements of E such that

$$R_x \leqq y$$

then according to Axiom A^* we must have either

$$x \leqq 0$$

or

$$x \parallel 0.$$

In the second case we would have

$$\lambda(y \vee 0) = (\lambda y) \vee 0 \leqq x \vee 0$$

(see [1]), where $y \vee 0 > 0$, for all $\lambda > 0$ which contradicts Axiom A^*. Lemma 4.2 is proved.

Let $R = \{r\}$ be an arbitrary set and G a linear system of real functions $g(r)$ defined on R. We introduce a *generalized natural ordering* of G: a subset $P \subset R$ is fixed and $g_1 < g_2$ denotes that $g_1(p) \leqq g_2(p)$ for all $p \in P$ and $g_1(p_0) < g_2(p_0)$ for at least one $p_0 \in P$.

We call P the *fundamental subset* of R.

We assume that G satisfies Axiom III.

THEOREM 4.1. *G satisfies Axiom A^*. G is G_A if and only if $g = 0$ is the unique function that vanishes identically on the fundamental subset $P \subset R$.*

Conversely, every E_{A^*} is isomorphic to a linear system of real functions $G = \{g\}$ defined on some set R with a generalized natural ordering. The functions $g \in G$ that correspond to the elements of the subsystem $W \subset E$ are those that are equal to 0 identically on the fundamental subset $P \subset R$.

PROOF. If $g_1 \in G$, $g_2 \in G$ satisfy the inequality

$$\lambda g_1 \leqq g_2$$

for any positive λ then necessarily

$$g_1(p) \leqq 0,$$

whenever $p \in P$. Consequently, the following cases are possible:

1) $g_1 = 0$ if $g_1(r) \equiv 0$ $(r \in R)$;

2) $g_1 < 0$ if there exists a $p_0 \in P$ such that $g_1(p_0) < 0$;

3) $g_1 \parallel 0$ if $g_1(p) \equiv 0 (p \in P)$ and $g_1(r_0) \neq 0$ for at least one $r_0 \in R \dot{-} P$. We see that Axiom A^* is fulfilled.

G satisfies Axiom A if and only if the third case $(g_1 \parallel 0)$ is excluded, for which the following condition is necessary and sufficient: there exist no $g \in G$ such that $g \neq 0$ and $g(p) \equiv 0$ $(p \in P)$.

Let E be an arbitrary system satisfying Axiom A^*. Suppose that

$$B = \{x_0, x_1, \cdots, x_\nu, \cdots\}, \qquad x_\nu \in E, \qquad \nu < \vartheta,$$

is a transfinite sequence of elements such that

1) $E = L(B)$,

2) for every ordinal $\nu < \vartheta$: $x_\nu \notin L(\{x_\mu\})$, where $0 \leqq \mu < \nu$,

3) there exists an ordinal $\vartheta' < \vartheta$ such that $W = L(\{x_\nu\})$, where $0 \leqq \nu < \vartheta'$.

Denote by P the set of all positive linear functions p on E and by Q the set of such linear functions q on E that

$$q(x_\nu) = 0,$$

whenever $\vartheta' \leqq \nu < \vartheta$. It follows from the definition of W that $P \cap Q$ is empty.

We introduce the notation R for $P \cup Q$. To every $x \in E$ we let correspond the function $g_x(r)$ defined on $R = \{r\}$,

$$x \to g_x(r) = r(x).$$

We have evidently

$$\alpha x + \beta y \to g_{\alpha x+\beta y}(r) = r(\alpha x + \beta y) = \alpha r(x) + \beta r(y) = \alpha g_x(r) + \beta g_y(r).$$

Introduce the generalized natural ordering of G determined by the fundamental subset $P \subset R$. We shall prove that $x > 0$ is equivalent to $g_x > 0$. It follows from Theorem 2.5 that $x > 0$ ($x < 0$) implies $g_x > 0$ (resp. $g_x < 0$). Suppose that $x \,||\, 0$. If $x \notin W$ then there exist such $p_1 \in P$, $p_2 \in P$ that $g_x(p_1) = p_1(x) > 0$, $g_x(p_2) = p_2(x) < 0$. If $x \in W$ then $g_x(p) \equiv p(x) \equiv 0$ ($p \in P$), but for at least one $q_0 \in Q$: $g_x(q_0) = g_0(x) \neq 0$. In both cases $g_x \,||\, 0$.

Incidentally, we have proved that the correspondence $x \to g_x(r)$ is biunivoque. Theorem 4.1 is proved.

Let $\tilde{E}$ be the factor system $E \ominus W$—the system of classes $\tilde{x} = x + W$, where $x \in E$. $\tilde{E}$ is a linear system. We define an ordering of $\tilde{E}$ as follows:

$\tilde{x}_1 < \tilde{x}_2$ if there exist such $x_1 \in \tilde{x}_1$, $x_2 \in \tilde{x}_2$ that $x_1 < x_2$.

Applying Theorem 4.1 it is easy to see that this is an actual ordering of $\tilde{E}$, i.e. the relation of order defined between the classes $\tilde{x}_1$, $\tilde{x}_2$ does not depend on the choice of the representatives x_1, x_2.

THEOREM 4.2. *$\tilde{E}$ is an Archimedean system.*

THEOREM 4.3. *If E is the direct sum of W and a subsystem $E_1 \subset E$, i.e. for every $x \in E$ there exists the unique representation*

$$x = z + y, \qquad z \in W, \qquad y \in E_1,$$[5]

then E_1 is an Archimedean system and the ordering of E can be re-defined as follows: $x_1 < x_2$ if for $x_1 = z_1 + y_1$, $x_2 = z_2 + y_2$ $y_1 < y_2$.

One can easily prove these theorems applying Theorem 4.1.

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[5] This is not the case in general.

§5. Closed systems

A very important class of partially ordered linear systems is that of closed systems.

Axiom C guarantees the existence of the l.u.b. for every set $M \subset E$ that has an u.b. Suppose that a set $M \subset E$ has a l.b. Then the set $N(M)$ is not empty and has u.b.'s (each $x \in M$ is one of them). Consequently there exists $\sup N(M)$ which is evidently the g.l.b. of M. We see that the set M has $\inf M$ if it has a l.b.

The following lemma will be useful.

LEMMA 5.1. *Let E be E_c. For any two subsets $M_1 \subset E$, $M_2 \subset E$ that have u.b.'s and $\lambda > 0$*

$$\sup (M_1 + M_2) = \sup M_1 + \sup M_2,$$

$$\sup (\lambda M_1) = \lambda \sup M_1.^{6}$$

PROOF. Since

$$M_1 + M_2 \leqq \sup M_1 + \sup M_2,$$

we have

$$\sup (M_1 + M_2) \leqq \sup M_1 + \sup M_2.$$

On the other hand

$$M_1 + M_2 \leqq \sup (M_1 + M_2)$$

implies

$$M_1 \leqq \sup (M_1 + M_2) - M_2,$$

$$\sup M_1 \leqq \sup (M_1 + M_2) - M_2,$$

$$M_2 \leqq \sup (M_1 + M_2) + \sup M_1,$$

$$\sup M_2 \leqq \sup (M_1 + M_2) - \sup M_1$$

and

$$\sup M_1 + \sup M_2 \leqq \sup (M_1 + M_2).$$

The proof of the second equality is not more difficult.

THEOREM 5.1. *Every closed system is Archimedean.*

PROOF. Suppose that E is E_c and E contains such elements x and y that

$$R_x \leqq y.$$

If $z = \sup R_x$ then by Lemma 5.1 for an arbitrary $\lambda > 0$

$$\lambda z = \sup (\lambda R_x).$$

---

[6] If E is not closed then these equalities hold, whenever the right sides exist.

Since $\lambda R_x = \overset{\cdot}{R}_x$, we have

$$\lambda z = z,$$

whence

$$z = 0.$$

Consequently

$$R_x \leqq 0$$

and in particular

$$x \leqq 0,$$

as was to be proved.

Let E be an arbitrary partially ordered linear system. We say that the partially ordered linear system E_1 is the *extension* of E if E is isomorphic to a subsystem $E' \cap E_1$. If E_1 is closed then we say that E_1 is a *closed extension* of E.

The problem of the existence of closed extensions of partially ordered linear systems has been studied by A. Youdine [6]. The following theorem contains the solution of the problem from another point of view.

THEOREM 5.2. *The necessary and sufficient condition for the existence of closed extensions of a partially ordered linear system E is that E be Archimedean. If E is E_A then among all closed extensions E_1 of E there exists a minimal one $\hat{E}$ such that every other E_1 is the extension of $\hat{E}$.*

PROOF. The necessity of the condition follows from Theorem 5.1 and the cogradiency of Axiom A.

In order to prove the sufficiency of the condition we shall use the results obtained by H. M. MacNeille who extended the Dedekind's method to arbitrary partially ordered sets [*F*].

Suppose that E is E_A and let $\hat{E}$ be the totality of all "cuts" (S, T) in E. The cut (S, T) is the pair of subsets $S \subset E$, $T \subset E$ such that

1) S, T are not empty,

2) $S = N(T)$ and $T = H(S)$.

The ordering of $\hat{E}$ is defined as follows:

$$(S, T) \leqq (S', T') \text{ if } S \subset S' \text{ (consequently } T \supset T').$$

The partially ordered set $\hat{E}$—we call it *closure* of E—possesses the following properties:

1. $\hat{E}$ is closed, i.e. it satisfies Axiom C.

2. The correspondence

$$x \to (S_x, T_x),$$

where $S_x = N(x)$, $T_x = H(x)$, is an isomorphism[7] between $E = \{x\}$ and the subset $E' \subset \hat{E}$ consisting of cuts of such kind.

[7] Since algebraic operations are not yet defined in $\hat{E}$, we understand here by *isomorphism* simply an ordering-preserving one-to-one correspondence.

3. If a subset $M \subset E$ has the l.u.b. (g.l.b.) $a \in E$ then the l.u.b. (resp. g.l.b.) of the set $M' \subset \hat{E}$ corresponding to M and considered as the subset of $\hat{E}$ is the cut (S_a, T_a) corresponding to a.

4. Every closed partially ordered set E_1 that contains a subset E_2 isomorphic to E contains the subset $\hat{E}_2 \supset E_2$ isomorphic to $\hat{E}$. If

$$x \to \varphi(x) \in E_2,$$

$$\xi \to \psi(\xi) \in \hat{E}_2,$$

where $x \in E$, $\xi \in \hat{E}$, are these isomorphisms, then for $\xi = (S_x, T_x)$

$$\varphi(x) = \psi(\xi).$$

We shall consider $\hat{E}$ as consisting of some elements ξ set into a one-to-one correspondence with the cuts (S, T) which will be denoted by

$$\xi \sim (S, T)$$

or

$$\xi \sim S,$$

where S is the lower class of the cut (S, T). The elements ξ corresponding to the "rational" cuts (S_x, T_x) will be identified with the elements $x \in E$. $\xi \leqq \xi'$ denotes that $S \subset S'$ if $\xi \sim S$, $\xi' \sim S'$.

Denote by $\mathfrak{S}$ and $\mathfrak{T}$ the systems of all sets S and T respectively. We shall use the following properties of the sets $S \in \mathfrak{S}$ (to everyone of them corresponds the dual property of the sets $T \in \mathfrak{T}$). The properties 1^0–4^0 hold for every partially ordered set E, the other ones deal with algebraic operations in E.

1^0. The sets $S \in \mathfrak{S}$ can be characterized by the following properties: S is not empty, has an u.b. and

$$S = N(H(S)).^8$$

2^0. If $x \in S$ and $x' < x$ then $x' \in S$.

3^0. If $\{S_\alpha\}$ is an arbitrary subsystem of $\mathfrak{S}$ and the intersection $S_0 = \bigcap_\alpha S_\alpha$ is not empty then $S_0 \in \mathfrak{S}$.

4^0. For every non-empty set $M \subset E$ that has an u.b. there exists the minimal set of the system $\mathfrak{S}$—denote it by $[M]$—that contains M and $[M] = N(H(M))$.

5^0. If $x \in E$ and $\lambda > 0$ then for every $S \in \mathfrak{S}$ the sets $S + x$ and λS belong to $\mathfrak{S}$.

6^0. If $S \in \mathfrak{S}$, $T \in \mathfrak{T}$ and $\lambda < 0$ then $\lambda S \in \mathfrak{T}$ and $\lambda T \in \mathfrak{S}$.

7^0. If the subset $M \subset E$ has an u.b. then $[M + y] = [\mathfrak{M}] + y$ and $[\lambda M] = \lambda[M]$, whenever $y \in E$ and $\lambda > 0$.

8^0. Every $S \in \mathfrak{S}$ is a convex set.

The property 1^0 can be easily deduced from the definition of the cut.

2^0 is evident.

---

[8] It is obvious that for every $M \subset E$ $M \subset N(H(M))$.

$\mathfrak{S}$ is isomorphic to $\hat{E}$ (if $S_1 \leqq S_2$ denotes that $S_1 \subset S_2$) and the closedness of $\hat{E}$ is based upon 3^0 which therefore is worth proving. Suppose that S_0 is not empty. Then it has an u.b., since $S_0 \subset S_\alpha$ and S_α has one. For every S_α

$$S_0 \subset S_\alpha = N(H(S_\alpha))$$

and consequently

$$H(S_0) \supset H(S_\alpha),$$

whence

$$N(H(S_0)) \subset N(H(S_\alpha)) = S_\alpha .$$

Since S_α was arbitrary, we have

$$N(H(S_0)) \subset S_0 .$$

On the other hand it is evident that

$$N(H(S_0)) \supset S_0 .$$

According to 1^0 $S_0 \in \mathfrak{S}$.

The first part of 4^0 follows immediately from 3^0: $[M]$ is the intersection of all S that contain M. Let us prove that $[M] = N(H(M))$. Consider the sets $[M] = \bigcap_{S \supset M} S$ and $S' = N(H(M))$. They both belong to $\mathfrak{S}$. Since

$$M \subset S',$$

we have

$$[M] \subset S'.$$

On the other hand

$$M \subset [M],$$

whence

$$H(M) \supset H([M])$$

and

$$S' = N(H(M)) \subset N(H([M])) = [M].$$

Consequently

$$[M] = N(H(M)).$$

5^0 and 6^0 can be easily proved with the use of 1^0.

If M has an u.b. and $y \in E$ then

$$[M] + y \supset M + y.$$

Since $[M] + y$ belongs to $\mathfrak{S}$ by 5^0, we have

$$[M] + y \supset [M + y].$$

Suppose that there exists such $x \in [M] + y$ that $x \bar{\in} [M + y]$. Since

$$x = a + y,$$

where $a \in [M]$, we would have

$$a \bar{\in} [M + y] - y.$$

The set $[M + y] - y$ belongs to $\mathfrak{S}$ and contains M. Consequently

$$[M + y] - y \supset [M] \supset a$$

and that contradicts our hypothesis. The second part of 7^0 can be proved in a similar way.

We now prove 8^0. Let x_1, x_2 be arbitrary elements of the set S. Then

$$x_1 \leqq T = H(S),$$

$$x_2 \leqq T = H(S),$$

i.e.

$$x_1 \leqq y,$$

$$x_2 \leqq y,$$

whenever $y \in T$. If λ and μ are non-negative numbers with $\lambda + \mu = 1$ then

$$\lambda x_1 \leqq \lambda y,$$

$$\mu x_2 \leqq \mu y,$$

and

$$\lambda x_1 + \mu x_2 \leqq \lambda y + \mu y = (\lambda + \mu) y = y.$$

Since $y \in T$ was arbitrary,

$$\lambda x_1 + \mu x_2 \leqq T$$

or

$$\lambda x_1 + \mu x_2 \in N(T) = S,$$

as was to be proved.

LEMMA 5.1. *For every cut (S, T) in E there exists inf $(T - S)$ which is equal to 0 if and only if E is E_A.*

PROOF. Without restricting generality we can suppose that $0 \in A$. If it were not so, we could "translate" the cut taking $(S - x_0, T - x_0)$ instead of (S, T), where $x_0 \in S$.

It is evident that $T - S \geqq 0$. Suppose that $T - S \geqq z$. In order to prove that 0 is the g.l.b. of $T - S$, we must show that $z \leqq 0$.

$$T - S \geqq z$$

implies

$$S + z \leqq T$$

or

$$S + z \subset S.$$

It is easy to see that

$$S + nz \subset S$$

for every integer n. Since $0 \in S$, we have in particular

$$0 + \{nz\} \subset S,$$

$$\{nz\} \subset S,$$

$$\{nz\} \leqq T.$$

It follows from Remark 1.2 that

$$R_z \leqq T,$$

whence by Axiom A

$$z \leqq 0.$$

If E is not E_A, i.e. E contains an element z positive or incomparable with 0 such that R_z has an u.b., then it is not difficult to see that the cut (S, T), where $S = [R_z]$, does not possess the property stated in Lemma 5.1.

After these preliminaries we can prove Theorem 5.2. We shall show that it is possible to define the algebraic operations in $\hat{E}$ in such way that $\hat{E}$ converts into a partially ordered linear system containing E with the a priori given operations as its subsystem.

DEFINITION O. *If* $\xi_1 \sim S_1$, $\xi_2 \sim S_2$ *then we put*

$$\xi_1 + \xi_2 \sim [S_1 + S_2].$$

If $\xi \sim (S, T)$ *then*

$$\lambda\xi \sim (\lambda S, \lambda T) \qquad \text{if } \lambda > 0,$$

$$\lambda\xi \sim (N(0), H(0)) = 0 \qquad \text{if } \lambda = 0,$$

$$\lambda\xi \sim (\lambda T, \lambda S) \qquad \text{if } \lambda < 0.$$

$\hat{E}$ with the operations just defined is a linear system. Let

$$\xi \sim S_1, \qquad \eta \sim S_2, \qquad \zeta \sim S_3$$

be arbitrary elements of $\hat{E}$.

I. $(\xi + \eta) + \zeta = \xi + (\eta + \zeta)$.

According to Definition O

$$(\xi + \eta) + \zeta \sim [[S_1 + S_2] + S_3],$$

$$\xi + (\eta + \zeta) \sim [S_1 + [S_2 + S_3]].$$

Since

$$S_1 + S_2 \subset [S_1 + S_2],$$

we have

$$S_1 + S_2 + S_3 \subset [S_1 + S_2] + S_3,$$

whence

$$[S_1 + S_2 + S_3] \subset [[S_1 + S_2] + S_3].$$

On the other hand, if

$$x \in H(S_1 + S_2 + S_3),$$

i.e.

$$x \geqq S_1 + S_2 + S_3,$$

then

$$x - S_3 \geqq S_1 + S_2$$

and by 4^0

$$x - S_3 \geqq [S_1 + S_2],$$
$$x \geqq [S_1 + S_2] + S_3$$
$$x \geqq [[S_1 + S_2] + S_3].$$

We see that

$$H(S_1 + S_2 + S_3) \subset H([S_1 + S_2] + S_3]),$$

whence

$$[S_1 + S_2 + S_3] = N(H(S_1 + S_2 + S_3))$$
$$\supset N(H([[S_1 + S_2] + S_3])) = [[S_1 + S_2] + S_3].$$

Consequently

$$[[S_1 + S_2] + S_3] = [S_1 + S_2 + S_3].$$

Analogously we can obtain

$$[S_1 + [S_2 + S_3]] = [S_1 + S_2 + S_3],$$

whence

$$[[S_1 + S_2] + S_3] = [S_1 + [S_2 + S_3]]$$

or

$$(\xi + \eta) + \zeta = \xi + (\eta + \zeta).$$

II. $\xi + \eta = \eta + \xi$.

We have, indeed, $S_1 + S_2 = S_2 + S_1$, whence $[S_1 + S_2] = [S_2 + S_1]$.

III. $0 \sim N(0)$ is the null-element of $\hat{E}$, i.e. $\xi + 0 = \xi$ for every $\xi \in \hat{E}$ or $[S + N(0)] = S$.

If $x \in S$, $y \in N(0)$ then $x + y \leqq x$. According to 2^0 $x + y \in S$. We see that

$$S + N(0) \subset S,$$

whence

$$[S + N(0)] \subset S.$$

Conversely, since $0 \in N(0)$, we have

$$S = S + 0 \subset S + N(0) \subset [S + N(0)].$$

IV. For every $\xi \in \hat{E}$ there exists such $-\xi \in \hat{E}$ that $\xi + -\xi = 0$.

Let $\xi \sim (S, T)$. We shall show that

$$-\xi = (-1)\xi \sim (-T, -S).$$

According to Definition O

$$\xi + -\xi \sim [S + (-T)] = [S - T].$$

Since $S - T \leqq 0$ or $S - T \subset N(0)$, we have

$$[S - T] \subset N(0).$$

Lemma 5.1 gives

$$\inf (T - S) = 0,$$

whence

$$\sup (S - T) = 0.$$

We obtain $x \geqq 0$, whenever $x \geqq S - T$, i.e.

$$H(S - T) \subset H(0),$$

whence by 1^0 and 4^0

$$[S - T] = N(H(S - T)) \supset N(N(0)) = N(0).$$

We obtain

$$[S - T] = N(0)$$

or

$$\xi + -\xi = 0.$$

V. $1 \cdot \xi = \xi$,

that is obvious.

VI. $\lambda(\mu\xi) = (\lambda\mu)\xi$ for arbitrary real numbers λ, μ and $\xi \sim (S, T)$.

If either λ or μ is equal to 0 then the equality is obvious. Consider, for instance, the case: $\lambda < 0$, $\mu > 0$. Then $\lambda\mu < 0$ and by Definition O

$$\mu\xi \sim (\mu S, \mu T),$$

$$\lambda(\mu\xi) \sim (\lambda(\mu T), \lambda(\mu S)),$$

$$(\lambda\mu)\xi \sim ((\lambda\mu)T, (\lambda\mu)S).$$

We see that $\lambda(\mu\xi)$ and $(\lambda\mu)\xi$ correspond to the same cut, i.e. $\lambda(\mu\xi) = (\lambda\mu)\xi$. One can consider the other cases analogously.

VII. $(\lambda + \mu)\xi = \lambda\xi + \mu\xi$, where $\xi \sim S$ and λ, μ are arbitrary.

If one of the numbers λ, μ is equal to 0, then the equality is obvious. Consider the case: $\lambda > 0$, $\mu > 0$. According to Definition O

$$(\lambda + \mu)\xi \sim (\lambda + \mu)S,$$

$$\lambda\xi + \mu\xi \sim [\lambda S + \mu S].$$

It is evident that

$$(\lambda + \mu)S \subset \lambda S + \mu S.$$

Conversely, if $x \in \lambda S + \mu S$, i.e.

$$x = \lambda x_1 + \mu x_2,$$

where $x_1 \in S$, $x_2 \in S$, then

$$x = (\lambda + \mu)\left(\frac{\lambda}{\lambda + \mu} x_1 + \frac{\mu}{\lambda + \mu} x^2\right).$$

According to 8^0

$$x_3 = \frac{\lambda}{\lambda + \mu} x_1 + \frac{\mu}{\lambda + \mu} x_2 \in S$$

and

$$x = (\lambda + \mu)x_3 \in (\lambda + \mu)S.$$

We obtain

$$\lambda S + \mu S \subset (\lambda + \mu)S,$$

consequently

$$[\lambda S + \mu S] = \lambda S + \mu S = (\lambda + \mu)S.$$

Using I, II, IV and VI we can reduce the other cases to the one just considered.

VIII. $\lambda(\xi + \eta) = \lambda\xi + \lambda\eta$.

Let $\xi \sim (S_1, T_1)$, $\eta \sim (S_2, T_2)$. The case: $\lambda = 0$, is trivial. Suppose that $\lambda > 0$. We have

$$\lambda(\xi + \eta) \sim \lambda[S_1 + S_2],$$

$$\lambda\xi + \lambda\eta \sim [\lambda S_1 + \lambda S_2].$$

It is evident that

$$\lambda S_1 + \lambda S_2 = \lambda(S_1 + S_2),$$

whence by 5^0

$$[\lambda S_1 + \lambda S_2] = [\lambda(S_1 + S_2)] = \lambda[S_1 + S_2].$$

The case: $\lambda = -1$, follows essentially from IV. The case: $\lambda < 0$, can be considered as the combination of the previous ones.

Suppose now that

$$\xi \sim S_1, \qquad \eta \sim S_2, \qquad \zeta \sim S_3$$

and $\xi \leqq \eta$, i.e. $S_1 \subset S_2$. Then

$$S_1 + S_3 \subset S_2 + S_3,$$

whence

$$[S_1 + S_3] \subset [S_2 + S_3]$$

or

$$\xi + \zeta \leqq \eta + \zeta.$$

If $\lambda > 0$ then

$$S_1 \subset S_2$$

implies

$$\lambda S_1 \subset \lambda S_2.$$

Since $\lambda\xi \sim \lambda S_1$, $\lambda\eta \sim \lambda S_2$, we obtain

$$\lambda\xi \leqq \lambda\eta.$$

We have proved that $\hat{E}$ satisfies Axioms I and II. Axiom III is also fulfilled, because $\hat{E}$ being closed is a lattice and for any two $\xi \in \hat{E}$, $\eta \in \hat{E}$ there exists $\eta = \xi \vee \eta$.

The operations in $\hat{E}$ introduced by Definition O being applied to the elements of E (we identify E with the set of "rational" cuts in E) coincide with the *a priori* given operations in E. This statement follows immediately from the evident equalities

$$N(x) + N(y) = N(x + y),$$

$$N(\lambda x) = \begin{cases} \lambda N(x), & \lambda > 0, \\ N(0), & \lambda = 0; \\ \lambda H(x), & \lambda < 0, \end{cases}$$

where x, y are arbitrary elements of E.

Suppose that the algebraic operations are defined in E so that E becomes a subsystem of $\hat{E}$. It follows from Lemma 5.1 that the operations in $\hat{E}$ necessarily coincide with those introduced in Definition O. This proves Theorem 5.2.

REMARK 5.1. The main result and the methods of this paragraph slightly modified hold for partially ordered groups. Axiom A for the multiplicative group $G = \{a\}$ can be formulated as follows:

If for an element $a \in G$ the set $\{a^n; n = 1, 2, \cdots\}$ has an u.b. then $a \leqq e$, where e is the unit.

§6. Systems E_{B^*}

Let E be E_{B^*}. The following lemma states some characteristic properties of axial elements.

LEMMA 6.1. *Each of the following conditions is necessary and sufficient in order that $u \in E$ be an axial element of E:*

(α) For every $x \in E$ the inequalities

$$-\lambda u \leqq x \leqq \lambda u$$

hold, whenever $\lambda > 0$ is sufficiently large [8].

(β) For every $x \in E$ the inequalities

$$-u \leqq \mu x \leqq u$$

hold, whenever $\mu > 0$ is sufficiently small.

PROOF. If u is an axial element of E then for every $x \in E$

$$x = \lambda' u - x',$$

$$-x = \lambda'' u - x'',$$

where $\lambda' \geqq 0$, $\lambda'' \geqq 0$, $x' \in E^+$, $x'' \in E^+$. Taking $\lambda \geqq \max\{\lambda', \lambda''\}$, we have

$$-\lambda u \leqq x \leqq \lambda u.$$

The converse is obvious.

(α) and (β) are evidently equivalent.

REMARK 6.1. Every E_{B^*} contains an infinity of axial elements. It is evident that if u is an axial element of E and $v > u$ then v is also an axial element.

Lemma 6.1 gives

LEMMA 6.2. *Let u be an axial element of E and $v > 0$. Then v is also an axial element of E if and only if*

$$\lambda u \leqq v,$$

whenever $\lambda > 0$ is sufficiently small.

THEOREM 6.1. *E is E_{B^*} if and only if E^+ contains a bounded set V such that every $x \in E^+$ can be represented in the form $x = \lambda y$, where $\lambda \geqq 0$, $y \in V$.*

PROOF. If E is E_{B^*} then we put

$$V = E^+ \cap N(u),$$

where u is an axial element of E.

Conversely, if E^+ contains such V then every $u \geqq V$ is an axial element.

THEOREM 6.2. *A weakly Archimedean system E is E_{B^*} if and only if the factor system $\tilde{E} = E \ominus W$ is $\tilde{E}_{B^*}$.*

The proof follows immediately from the definition of $\tilde{E}$ (§4).

THEOREM 6.3. *An Archmidean system E is E_{B^*} if and only if its closure $\hat{E}$ is $\hat{E}_{B^*}$. If it is so, then the axial element of $\hat{E}$ can be chosen in E.*

It is easy to verify these statements using Remark 6.1.

THEOREM 6.4. *If E is E_{B^*,C^*} then every axial element $u \in E$ is a unit in the sense of H. Freudenthal, i.e. $u \wedge x > 0$, whenever $x > 0$.*

PROOF. Assume the contrary. We would have an element $x > 0$ such that

$$u \wedge x = 0.$$

Then for every $\lambda > 0$

$$(\lambda u) \wedge x = 0$$

and

$$\lambda u + x = (\lambda u) \vee x \qquad [1]$$

Owing to Lemma 6.1 $\lambda > 0$ can be chosen such that

$$\lambda u \geqq x.$$

Therefore

$$(\lambda u) \vee x = \lambda u$$

and we obtain

$$\lambda u + x = \lambda u,$$

$$x = 0,$$

which contradicts our hypothesis.

REMARK 6.2. The converse is not true: there exist lattices that have units and do not contain axial elements at all.

Let u be a fixed axial element, $y \in E$ and α an arbitrary positive number. Consider the set $U_\alpha(y)$ consisting of the element x with

$$-(\alpha - \epsilon)u \leqq y - x \leqq (\alpha - \epsilon)u,$$

where $\epsilon > 0$ depends on x. Every such set we call *interval* in E. Introduce the notation $\mathfrak{A}$ for the totality of all intervals in E.

THEOREM 6.5. *A topology can be defined in E satisfying the conditions:*

1. *E converts into a T_1-space* [9];
2. *algebraic operations in E are continuous;*
3. *$\mathfrak{A}$ is the basis of the space E;*

if and only if E is homogeneous. If E is E_R then the needed topology can be generated by a norm.

PROOF. Suppose that E is not homogeneous, i.e. E contains an element $z \neq 0$ with

$$x_1 \leqq D_z \leqq x_2 .$$

Then for every $\mu > 0$

$$\mu^2 x_1 \leqq \mu^2 D_z \leqq \mu^2 x_2 .$$

Since $\mu^2 D_z = D_z$,

$$\mu^2 x_1 \leqq D_z \leqq \mu^2 x_2 .$$

Putting μ small enough in order that

$$-u \leqq \mu x_i \leqq u \qquad (i = 1, 2),$$

where u is a fixed axial element of E, we obtain

$$-\mu u \leqq \mu^2 x_i \leqq \mu u$$

and

$$-\mu u \leqq D_z \leqq \mu u .$$

Since in these inequalities we can take μ as small as we please, D_z and z in particular belong to every $U_\mu(0)$. Therefore 1. cannot be fulfilled.

If E is E_H then we define for every $x \epsilon E$

$$|| x ||_u = \sup \{\mu; -u \leqq \mu x \leqq u\} \qquad [8].$$

For every $x \epsilon E$ $|| x ||_u$ is finite. It is an immediate application of Axiom H. Lemma 6.1 gives that $|| x ||_u = 0$ only if $x = 0$. Furthermore it is evidently

$$|| \lambda x ||_u = | \lambda | \, || x ||_u ,$$

$$|| x + y ||_u \leqq || x ||_u + || y ||_u .$$

We see that $|| x ||_u$ possesses the properties of the norm, we call it *u-norm*. It is evident that $U_\alpha(y)$ is the (open) α-sphere with the center y in the sense of u-norm. This completes the proof of the theorem.

In the following we shall understand by "u-completeness", "u-convergence" etc. the corresponding notions based on the u-norm. It is evident that if u and v are different axial elements of E then u-norm and v-norm are topologically equivalent.

REMARK 6.3. If E is a u-normed lattice then

$$|| x ||_u = \sup \{\mu > 0; \mu | x | \leqq u\} = \inf \{\lambda > 0; | x | \leqq \lambda u\},$$

where

$$| x | = x \vee (-x) \qquad [2].$$

It is easy to deduce from this remark the following

LEMMA 6.3. *If E is u-normed and closed, x is the u-limit of a sequence $\{x_n\} \subset E$ then $\{x_n\}$ converges to x in the sense of L. Kantorovitch* [2].

THEOREM 6.6. *If E is u-normed and closed then E is u-complete.*

PROOF. Suppose that $\{x_n\}$ is such that for every $\epsilon > 0$

$$| x_m - x_n | \leqq \epsilon u,$$

whenever m and n are sufficiently large. According to [2], Theorem 20, $\{x_n\}$ has the limit x in the sense of L. Kantorovitch and the inequalities

$$-\epsilon u \leqq x_m - x_n \leqq \epsilon u$$

imply

$$-\epsilon u \leqq x - x_n \leqq \epsilon u,$$

that means x is the u-limit of $\{x_n\}$.

§7. Partially ordered Banach spaces

In this paragraph we consider partially ordered Banach spaces, actually those the unit sphere of which has an u.b. and a l.b.[9]

We assume that E is E_B.

LEMMA 7.1. *Every E_B is E_{B^*} and the u.b. of the unit sphere is an axial element of E_B.*

PROOF. E_B satisfies the condition of Theorem 6.1 if we put V equal to the intersection of E_B^+ with the unit sphere. Every u.b. of the unit sphere is an u.b. of V and consequently is an axial element of E_B.

The *a priori* given norm in E we call N-norm. The topological and metrical properties of E based on the N-norm such as "closure", "boundedness" etc. are called "N-closure", "N-boundedness" etc.

LEMMA 7.2. *Axiom B is equivalent to the following condition: E is a Banach space ordered in such a way that E^+ contains an N-inner point.*

PROOF. Denote by $S(y; \alpha)$ the (N-open) α-sphere with the center y. If E is E_B, i.e.

$$S(0; 1) \leqq y$$

then

$$y - S(0; 1) = y + S(0; 1) \geqq 0.$$

---

[9] Such is, for instance, the space C of continuous functions $x(t)$, $0 \leqq t \leqq 1$, with the natural ordering and the norm $|| x || = max | x(t) |$. On the contrary, the space $L_p (p \geqq 1)$ of measurable functions $x(t)$, $0 \leqq t \leqq 1$, with summable $| x(t) |^p$ does not satisfy Axiom B if $|| x || = \left(\int_0^1 | x(t) |^p \, dt \right)^{\frac{1}{p}}$ and $x < y$ denotes that $x(t) \leqq y(t)$ almost everywhere on $[0, 1]$ and $x(t) < y(t)$ on a set $E \subset [0, 1]$ of positive measure.

Since $y + S(0; 1) = S(y; 1)$, we obtain

$$S(y; 1) \geqq 0,$$

y is an N-inner point of E^+.

Conversely, if y is an N-inner point of E^+, i.e.

$$S(y; \epsilon) \geqq 0,$$

then

$$S(y; \epsilon) = y + S(0; \epsilon) = y + \epsilon S(0; 1) \geqq 0.$$

We obtain

$$S(0; 1) = -S(0; 1) \leqq \frac{1}{\epsilon} y,$$

as was to be proved.

Lemma 7.2 gives [10]

LEMMA 7.3. *If E is E_B then every positive linear function $p(x)$ on E is N-continuous.*

THEOREM 7.1. *E_B is Archimedean if and only if E^+ is N-closed.*

PROOF. Suppose that E is $E_{A,B}$. Let y be an arbitrary point of the N-boundary of E^+ and u an N-inner point of E^+. Since E^+ is convex, every

$$\alpha y + (1 - \alpha)u$$

with $0 \leqq \alpha < 1$ belongs to E^+, i.e.

$$\alpha y + (1 - \alpha)u \geqq 0,$$

whence

$$y + \frac{1 - \alpha}{\alpha} u \geqq 0.$$

By Lemma 3.2 this inequality holds for $\alpha = 1$ which gives

$$y \geqq 0.$$

The sufficiency of the condition can be deduced from Lemma 3.2 in an elementary way.

THEOREM 7.2. *Let E be E_B. If E is weakly Archimedean then every point y of the N-boundary of E^+ belongs either to E^+ or to W.*

PROOF. It follows from Lemma 7.3 that W is N-closed in $E_{A^*,B}$ and consequently $\tilde{E} = E \ominus W$ is a Banach space if we put $|| \tilde{x} || = \inf \{ || x ||; x \in \tilde{x} \}$. It is also easy to see that $\tilde{E}$ is $\tilde{E}_B$.

If $\tilde{y} = y + W$ then $\tilde{y} \in \tilde{E}^+$ (use Theorems 4.2 and 7.1). If $\tilde{y} = \tilde{0} = W$ then $y \in W$; $\tilde{y} \neq \tilde{0}$ implies $y \notin W$, $y \in E^+$, as was to be proved.

If E is $E_{B,H}$ then the u-norm can be defined in E. According to Axiom B every N-bounded subset of E is bounded which implies that the N-topology in

E is generally stronger than the u-topology. They are obviously equivalent if, conversely, every bounded subset of E is N-bounded. Since every bounded subset $M \subset E$ can be included in an interval of the system $\mathfrak{A}$, we thus obtain

THEOREM 7.3. *If E is a homogeneous E_B then the N-norm in E is topologically equivalent to the u-norm if and only if the intervals in E are N-bounded.*

We say that the N-norm is *non-decreasing* if

$$-y \leqq x \leqq y$$

implies

$$\| x \| \leqq \| y \|.$$

LEMMA 7.4. *The N- and u-norms are topologically equivalent if and only if the N-norm is topologically equivalent to a non-decreasing norm (in particular, is non-decreasing itself).*

PROOF. Obvious, since the u-norm is non-decreasing and any two non-decreasing norms are topologically equivalent.

According to the results obtained by M. Krein ([10] and [11]) the N-boundedness of the intervals in E is equivalent to the following condition imposed on the adjoint space $\bar{E}$:

Every linear functional $f \in \bar{E}$ can be represented

$$f = h_1 - h_2,$$

where h_1, h_2 are positive linear functionals, i.e. $h_i(x) \geqq 0$, whenever $x \in E^+$ $(i = 1,2)$.

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REFERENCES

[1] H. FREUDENTHAL, Proc. Kon. Akad. Amsterdam, XXXIX, No. 5 (1936).
[2] L. KANTOROVITCH, Rec. Math., 2(44):1 (1937).
[3] B. L. VAN DER WAERDEN, Moderne Algebra, Berlin (1930).
[4] G. BIRKHOFF, Proc. Cambridge Phil. Soc., 29 (1933).
[5] A. TYCHONOFF, Math. Annalen, 111 (1935).
[6] A. YOUDINE, Annals of Leningrad University, 12 (1941).
[7] H. M. MACNEILLE, Proc. Nat. Acad. U.S.A., 22 (1936).
[8] M. KREIN, S. KREIN, C. R. de l'Acad. Sc. U.R.S.S., XXVII, N5 (1940).
[9] P. ALEXANDROFF, E. HOPF, Topologie, I, Berlin (1935).
[10] N. AKHIESER, M. KREIN, On some problems in the theory of moments (in Russian), Kharkhov, (1938).
[11] M. KREIN, C. R. de l'Acad. Sc. U.R.S.S., XXVIII, N1 (1940).

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HOMOLOGY WITH LOCAL COEFFICIENTS

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1. Introduction

In a recent paper [16] the author has had occasion to introduce and use what he believed to be a new type of homology theory, and he named it *homology with local coefficients*. It proved to be the natural and full generalization of the Whitney notion of *locally isomorphic complexes* [18]. Whitney, in turn, credits the source of his idea to de Rham's *homology groups of the second kind in a non-orientable manifold* [13]. It has since come to the author's attention that homology with local coefficients is equivalent in a complex to Reidemeister's *Überdeckung* [10].

Since this new homology theory (which includes the old) seems to have such wide applicability, a complete review of the older theory is needed to determine to what extent and in what form its theorems generalize. The object of this paper is to make such a survey. The general conclusion is that all major parts of the older theory do extend to the new. In addition the newer theory fills in several gaps in the old. The most noteworthy of these is a full duality and intersection theory in a non-orientable manifold (§14).

For the sake of completeness, some of the results of Reidemeister have been included. The new approach and new definitions make for easier and more intuitive proofs. They lead also to results not obtained by Reidemeister. The most important is a proof of the topological invariance of all the homology groups obtained.[1] In addition developments are given of the subjects of multiplications of cycles and cocycles, chain mappings, continuous cycles, and Čech cycles.

Part I contains an abstract development of systems of local groups in a space entirely apart from their applications to homology. Any fibre bundle over a base space R [18] determines many such systems in R (one for each homology group, homotopy group, etc., of the fibre). These are invariants of the bundle. They should prove to be of some help in classifying fibre bundles.

Part II, which contains the extended homology theory, presupposes on the part of the reader a knowledge of the classical theory such as can be found in the books of Lefschetz [7] and Alexandroff-Hopf [1].

I. LOCAL GROUPS IN A SPACE

2. Notations

We shall be dealing throughout with an arcwise connected topological space R. For any point x of R, let F_x be the fundamental (Poincaré) group of R with

[1] It is not determined however whether or not the combinatorial invariant called "torsion" by Reidemeister [9] and its generalizations by Franz [4] and de Rham [14] are true topological invariants.

x as initial and terminal point. If A is a curve from x to y, the class of curves from x to y homotopic to A with end points fixed we shall denote by a symbol such as α_{xy}. Its inverse is denoted by α_{xy}^{-1} or α_{yx}. The elements of F_x are abbreviated α_x, β_x, etc. The class α_{xy} determines an isomorphism $F_x \to F_y$ (denoted by α_{xy}) defined by $\alpha_{xy}(\beta_x) = \alpha_{yx}\beta_x\alpha_{xy}$. In keeping with this notation, the product $\alpha_x\beta_x$ means the element of F_x obtained by traversing first a curve of the class α_x then one of β_x. As is well known, the combination of two isomorphisms $\beta_{yz}(\alpha_{xy}(\gamma_z))$ is the isomorphism $(\alpha_{xy}\beta_{yz})(\gamma_z)$.

3. Local groups

We shall say that we have *a system of local groups (rings) in the space R* if (1) for each point x, there is given a group (ring) G_x, (2) for each class of paths α_{xy}, there is given a group (ring) isomorphism $G_x \to G_y$ (denoted by α_{xy}), and (3) the result of the isomorphism α_{xy} followed by β_{yz} is the isomorphism corresponding to the path $\alpha_{xy}\beta_{yz}$.

It follows from the transitivity condition (3) that the identity path from x to x is the identity transformation in G_x. A further consequence is that the inverse of the isomorphism α_{xy} is α_{yx}. By (2), a closed path $\alpha_x \in F_x$ determines an automorphism of G_x. From (3) it follows that F_x is a group of automorphisms of G_x. The invariant subgroup of F_x acting as the identity on G_x is denoted F_x^1. Since, by (3),

$$\alpha_{xy}(\beta_x(\alpha_{yx}(g))) = (\alpha_{yx}\beta_x\alpha_{xy})(g), \qquad g \in G_y,$$

it follows that

$$\alpha_{xy}(\beta_x(g)) = [\alpha_{xy}(\beta_x)](\alpha_{xy}(g)), \qquad g \in G_x. \tag{3.1}$$

We shall say that the system $\{G_x\}$ is *simple* if every $F_x^1 = F_x$. If this happens for one x, it will be true for all. If $\{G_x\}$ is simple, the isomorphism α_{xy} is independent of the path from x to y. Choosing a fixed point o as origin, we find that each G_x is uniquely isomorphic to G_o. Thus the local system consists of one G_o and as many copies of G_o as there are points $x \neq 0$.

Two systems $\{G_x\}$, $\{H_x\}$ are said to be *isomorphic* if, for each x, there is an isomorphism ϕ_x of G_x onto H_x such that

$$\phi_y(\alpha_{xy}(g)) = \alpha_{xy}(\phi_x(g)), \qquad g \in G_x.$$

We shall deal only with properties of systems which are invariant under isomorphisms. In each case the proof of invariance is trivial and will be omitted.

It was proved in §2 that the collection $\{F_x\}$ is a system of local groups. It is simple if and only if it is abelian.

In some instances a system $\{G_x\}$ will consist of topological groups. The isomorphisms α_{xy} will then be continuous. In the following pages we shall omit continuity considerations whenever such are reasonably obvious.

We shall consistently attempt to reduce the study of a system to the study of what occurs at one point of R. As a first step we have

THEOREM 1. *If G is a group (ring), o a point of R, and ψ a homomorphism of F_o into the group of automorphisms of G, then there is one and only one system $\{G_x\}$ of local groups (rings) in R such that G_o is a copy of G and the operations of F_o in G_o are those determined by ψ.*

For each point $x \in R$, choose a class of paths λ_{ox}, choosing $\lambda_{oo} =$ identity. Let G_x be a group (ring) isomorphic to G. Associate this isomorphism with λ_{xo}. If α_{xy} is a path, we attach to it the isomorphism $G_x \to G_y$ defined by

$$\alpha_{xy}(g) = \lambda_{oy}[(\lambda_{ox}\alpha_{xy}\lambda_{yo})(\lambda_{xo}(g))].$$

The second operation is the automorphism of G attached to $\lambda_{ox}\alpha_{xy}\lambda_{yo} \in F_o$ by ψ. Since ψ is a homomorphism, the transitivity condition (3) holds.

If $\{G'_x\}$ is a local system, and ϕ an isomorphism of G'_o into G_o such that $\alpha\phi = \phi\alpha$ for $\alpha \in F_o$, map $G'_x \to G_x$ with the operation $\phi_x(g') = \lambda_{ox}(\phi(\lambda_{xo}(g')))$. It is easily verified that $\{\phi_x\}$ establishes an isomorphism between $\{G'_x\}$ and $\{G_x\}$. This proves the uniqueness and completes the theorem.

4. Automorphisms

Let $\{G_x\}$, $\{A_x\}$ be two systems of local groups, and suppose that A_x is a group of automorphisms of G_x in such a way that, for any α_{xy}, we have

$$\alpha_{xy}(a(g)) = \alpha_{xy}(a)(\alpha_{xy}(g)), \qquad a \in A_x,\ g \in G_x. \tag{4.1}$$

Then $\{A_x\}$ is called *a system of local automorphisms of* $\{G_x\}$. By (3.1), it follows that $\{F_x\}$ is such a system for any $\{G_x\}$.

Let A_x^1 be the subgroup of A_x acting as the identity on G_x. By (4.1), $\{A_x^1\}$ is a system of local groups. It follows that, under the natural isomorphisms $A_x/A_x^1 \to A_y/A_y^1$ induced by $A_x \to A_y$, $\{A_x/A_x^1\}$ is a system of local automorphisms of $\{G_x\}$.

If A is a group, and, for each x, A is a group of automorphisms of G_x such that

$$\alpha_{xy}(a(g)) = a(\alpha_{xy}(g)), \qquad a \in A,\ g \in G_x, \tag{4.2}$$

we shall call A *a group of uniform automorphisms of* $\{G_x\}$. If a system $\{A_x\}$ of local automorphisms is simple, then any one of its groups is, in a natural way, a group of uniform automorphisms of $\{G_x\}$. If F_x or F_x/F_x^1 is abelian, we have such a group of uniform automorphisms for any $\{G_x\}$.

As in Theorem 1, complete knowledge of a system of automorphisms is obtainable from knowledge of what occurs at a single point.

THEOREM 2. *Let A be a group of automorphisms of G with only the identity acting as such in G (i.e. $A^1 = 1$). Let o be a point of R, and let F_o be represented as a group of automorphisms of G in such a way that the automorphism $\alpha(a(\alpha^{-1}(g)))$ of G is in A for every $\alpha \in F_o$, $a \in A$. Then there is one and only one system $\{A_x\}$ of local automorphisms of $\{G_x\}$ such that the collection (F_o, G_o, A_o) is isomorphic to (F_o, G, A). $\{A_x\}$ is simple and therefore A is a group of uniform automorphisms of $\{G_x\}$ if and only if the automorphisms of F_o and A in G commute.*

By Theorem 1, the system $\{G_x\}$ is completely determined. By assumption

the automorphism $\alpha a \alpha^{-1}$ is a unique element of A. Thus each α determines an automorphism of A, and F_o is a group of such. By Theorem 1, the system $\{A_x\}$ of local groups is completely determined. For $a \in A_x$, $g \in G_x$, we choose a path α_{ox} and define

$$a(g) = \alpha_{ox}(\alpha_{xo}(a)(\alpha_{xo}(g))).$$

Clearly (4.1) will hold once we have proved the right side to be independent of α_{ox}. This is shown as follows.

$$\begin{aligned}\beta_{xo}(\alpha_{ox}(\alpha_{xo}(a)(\alpha_{xo}(g)))) &= (\alpha_{ox}\beta_{xo})(\alpha_{xo}(a)(\alpha_{xo}(g))) \\ &= [(\alpha_{ox}\beta_{xo})(\alpha_{xo}(a))(\alpha_{ox}\beta_{xo})^{-1}]((\alpha_{ox}\beta_{xo})(\alpha_{xo}(g))) \\ &= \beta_{xo}(a)(\beta_{xo}(g)).\end{aligned}$$

5. Operator rings

If G is an additive abelian group, the set H of all homomorphisms of G into itself forms a ring under the operations

$$(a + b)(g) = a(g) + b(g), \qquad (ab)(g) = a(b(g)), \qquad a, b \in H, g \in G.$$

A group A of automorphisms of G forms a multiplicative subgroup of H. It generates a subring A^* of H with unit = identity. If the symbol $a(g)$ be abbreviated by ag, this multiplication of $g \in G$ by the scalar $a \in A^*$ obeys the usual laws: $(a + b)g = ag + bg$, $(ab)g = a(bg)$, $1g = g$, $0g = 0$, $a(g + g') = ag + ag'$. One may therefore speak of linear combinations, linear independence, and bases in G relative to A.

We shall say that the system $\{A_x\}$ is *a system of operator rings for the abelian system* $\{G_x\}$ if, for each x, A_x is a ring of operators for G_x, and for each path α_{xy}, we have

$$\alpha_{xy}(ag) = \alpha_{xy}(a)\alpha_{xy}(g), \qquad a \in A_x, g \in G_x.$$

The analogue of Theorem 2 is proved with only slight modifications. If a system of operator rings is simple, we are led naturally to the concept of *a uniform operator ring for* $\{G_x\}$.

For any $\{G_x\}$, the system $\{F_x^*\}$ of group rings of $\{F_x\}$ is a system of operator rings. If F_x or F_x/F_x^1 is abelian, the group ring of F_o or factor ring thereof is a uniform operator ring for $\{G_x\}$.

6. Dual systems

Two abelian systems $\{G_x\}$, $\{H_x\}$ of local groups form a *pair* with respect to a third $\{K_x\}$ if, for each x, G_x and H_x form a pair with respect to K_x (i.e. a multiplication $gh = k$ is given which is linear in each factor) in such a way that, for each path α_{xy}, we have $\alpha_{xy}(gh) = \alpha_{xy}(g)\alpha_{xy}(h)$. Analogous to Theorem 1, we have

THEOREM 3. *Let G, H form a pair with respect to K, and let F_o be realized as a group of automorphisms of each of G, H, K in such a way that $\alpha(gh) = \alpha(g)\alpha(h)$ for $\alpha \in F_o$, $g \in G$, $h \in H$. Then there is one and only one set of systems $\{G_x\}$, $\{H_x\}$,*

$\{K_x\}$ such that the first two form a pair with respect to the third, and G_o, H_o, K_o and the automorphisms F_o form a set isomorphic to the given G, H, K, F_o.

By Theorem 1, there are unique systems $\{G_x\}$, $\{H_x\}$, $\{K_x\}$ corresponding to G, H, K and the operations of F_o in these groups. In the construction of all three systems let us use the same collection of paths λ_{ox} (see proof of Theorem 1). Under the isomorphism of G_x, H_x, K_x with G, H, K attached to λ_{ox} the multiplication in the latter groups carries over into a multiplication in the former. The relation $\alpha_{xy}(gh) = \alpha_{xy}(g)\alpha_{xy}(h)$ for $g \in G_x$, $h \in H_x$ is proved by using the property $\alpha(g'h') = \alpha(g')\alpha(h')$ of the closed path $\alpha = \lambda_{ox}\alpha_{xy}\lambda_{yo}$ where g', h' correspond to g, h under λ_{ox}. Any other allowable product which agrees with the constructed product in G_o, H_o will likewise agree in G_x, H_x since both products are invariant under the translation along λ_{xo}.

Of particular interest is the case of character groups.

THEOREM 4. *If K = real numbers* mod 1, *G = a discrete, or compact, or locally compact separable group, and F_o is realized in two arbitrary ways as a group of automorphisms of K_o and G_o respectively, then F_o can be realized in one and only one way as a group of automorphisms of the character group H of G satisfying $\alpha(gh) = \alpha(g)\alpha(h)$. Thus to given systems $\{K_x\}$, $\{G_x\}$ is attached a unique system $\{H_x\}$ of character groups.*

The character $\alpha(h)$ is defined to be the one with the value $\alpha(\alpha^{-1}(g)h)$ on any g. The remainder of the theorem is readily verified.

It is to be noted that K admits but one non-trivial automorphism, namely: $k \to -k$. Thus there are as many character systems of a given system $\{G_x\}$ as there are factor groups of F_o of order 2. If it should be desirable to have a *unique* character system, it would be natural to choose $\{K_x\}$ to be simple.

7. Local groups under mappings

Let R, R' be two arcwise connected spaces and ϕ a continuous map $R' \to R$. Let F, F' be their fundamental groups relative to base points o, o' such that $\phi(o') = o$. If $\{G_x\}$ is a system of local groups in R, then ϕ *induces* in R' a system $\{G'_y\}$ as follows. The group G'_y is chosen isomorphic to G_x where $x = \phi(y)$. The isomorphism is denoted by ϕ. The path α_{yz} in R' maps G'_y isomorphically on G'_z according to the rule

$$\alpha_{yz}(g) = \phi^{-1}(\phi(\alpha_{yz})(\phi(g))), \qquad g \in G'_y. \tag{7.1}$$

The transitivity condition is immediately verified.

The existence of the induced system is apparent in view of Theorem 1 and the fact that the homomorphism $F' \to F$ realizes F' as a group of automorphisms of G_o.

The induced system has numerous properties which we list without proofs.

(a) If $\{G_x\}$ is simple, so is $\{G'_y\}$.

(b). $\{G'_y\}$ is simple if and only if F' is mapped by ϕ on the subgroup F^1 of F leaving G_o fixed.

(c). If $\{G_x\}$, $\{H_x\}$ are paired relative to $\{K_x\}$, then likewise the induced systems in R'.

(d). If $\{A_x\}$ is a system of local automorphisms or operator rings for $\{G_x\}$, then likewise the induced systems in R'.

(e). If A is a group of uniform automorphisms or a uniform operator ring for $\{G_x\}$, it is also one for the induced system.

It follows from (b) that, if R' is the covering space of R corresponding to the subgroup F^1 of F, the induced system is simple. Thus any system in R can be considered as the continuous image of a simple system in some covering space.

It is natural to inquire under what circumstances a given system in R' is induced by one in R. For this it is necessary and sufficient that (1) the kernel $\bar{F}'$ of the homomorphism $F' \to F$ shall act as the identity on $G'_{o'}$, and (2) the map of the subgroup $F'/\bar{F}'$ of F into the group of automorphisms of $G'_{o'}$ shall admit a homomorphic extension to F.

8. Special local groups

Because of their importance in the work of Reidemeister, we shall discuss certain local systems based on the fundamental group F of R.

Let F^1 be an invariant subgroup of F, and let $\mathfrak{F} = F/F^1$. Let $\mathfrak{G}$ be an abelian group. Let G be the set of functions from $\mathfrak{F}$ to $\mathfrak{G}$. If two such functions are added by adding functional values at each element of $\mathfrak{F}$, G becomes an abelian group. A function $f \in G$ is called a *restricted* function if $f(\alpha) = 0$ except for a finite number of $\alpha \in \mathfrak{F}$. The restricted functions form a subgroup G' of G. The structure of G (G') can be described as the unrestricted (restricted) direct sum of as many copies of $\mathfrak{G}$ as there are elements of $\mathfrak{F}$. If $\mathfrak{G}$ = integers and $F^1 = 1$, then G' is the ordinary group ring of F.

If $\mathfrak{G}$ is a ring, we define a multiplication of two functions $f, g \in G'$ by

$$f \times g(\alpha) = \sum_\beta f(\alpha\beta^{-1})g(\beta), \qquad \alpha, \beta \in \mathfrak{F}.$$

Since the functions in G' are restricted, the sum is finite; thus G' is a ring. If $\mathfrak{G}$ = integers, this product is the usual one in the group ring.

The group F can be realized in three natural ways as a group of automorphisms of the groups G, G'. For any $\gamma \in F$, we define three operations on a function $f \in G$ (G') by

$$L_\gamma f(\alpha) = f(\gamma^{-1}\alpha), \qquad R_\gamma f(\alpha) = f(\alpha\gamma), \qquad T_\gamma f(\alpha) = f(\gamma^{-1}\alpha\gamma), \qquad \alpha \in \mathfrak{F}.$$

Then $L_\gamma f$, $R_\gamma f$ and $T_\gamma f$ are in G (G'), and are called the *left translation*, *right translation*, and *transform* of f by γ, respectively. It is easy to verify that L_γ, R_γ, T_γ are automorphisms of G (G'), and that $L_\gamma L_\delta = L_{\gamma\delta}$, $R_\gamma R_\delta = R_{\gamma\delta}$, and $T_\gamma T_\delta = T_{\gamma\delta}$. The subgroup acting as the identity for both L and R is F^1.

In the case that $\mathfrak{G}$ and therefore G' is a ring, the left and right translations are not *ring* automorphisms. However the transforms are: $T_\gamma(f \times g) = (T_\gamma f) \times (T_\gamma g)$.

It follows from Theorem 1 that, corresponding to each of the three ways that

F can act on G (G'), we have a system of local groups. These we denote by $\{G_x^L\}$, $\{G_x^R\}$, $\{G_x^T\}$, and similarly for G'. The last of these, $\{G_x'^T\}$, is a system of local rings, whenever $\mathfrak{G}$ is a ring.

Since $\mathfrak{G}$ is associative, $L_\gamma R_\beta f = R_\beta L_\gamma f$. Therefore, by Theorem 2, the left translations form a group of uniform automorphisms of $\{G_x^R\}$, and likewise for the right translations of $\{G_x^L\}$. Then, as in §5, the group ring of F is a ring of uniform operators for these systems.

Under the automorphism ϕ of G (G') defined by $\phi f(\alpha) = f(\alpha^{-1})$, we have $\phi L_\gamma f = R_\gamma \phi f$. Therefore, by means of ϕ, an isomorphism exists between $\{G_x^L\}$ and $\{G_x^R\}$.

In the work of Reidemeister on homotopy chains, $\mathfrak{G}$ = integers, $F^1 = 1$, so that G' is the group ring of F. The coefficients of the homotopy chains belong to the local groups $\{G_x'^R\}$ (these are not rings), and the ring of uniform operators is likewise G' where left translations are used. This distinction between the two usages of the group ring of F is necessary for a comprehension of the subject of homotopy chains.

II. The combinatorial theory

9. Chains with local coefficients

Let $\{G_x\}$ be a system of local abelian groups in a space R which is decomposed into a cell complex[2] K. A q-cell of K is denoted by σ^q, incidence by $\sigma < \sigma'$, and incidence numbers by $[\sigma^{q-1}:\sigma^q]$. We suppose as usual that

$$\sum_{\sigma^{q-1}} [\sigma^{q-2}:\sigma^{q-1}][\sigma^{q-1}:\sigma^q] = 0. \tag{9.1}$$

In each cell σ we choose a representative point $x(\sigma)$ and abbreviate the symbol $G_{x(\sigma)}$ by G_σ. A q-chain of K is a function[3] f attaching to each oriented q-cell σ an element $f(\sigma) \in G_\sigma$ with the property $f(-\sigma) = -f(\sigma)$. Chains are added by adding functional values. They then form a group isomorphic to the direct sum of the groups G_σ for all q-cells σ.

If $\sigma' < \sigma$, we may choose a path in the closure of σ joining $x(\sigma)$ to $x(\sigma')$ and obtain therefrom an isomorphism $G_\sigma \to G_{\sigma'}$ which is denoted by $h_{\sigma'\sigma}$. In order that $h_{\sigma'\sigma}$ shall be independent of the path, we postulate that the closure of each cell is simply connected. A second consequence of this and the transitivity condition is

$$h_{\sigma''\sigma'} h_{\sigma'\sigma} = h_{\sigma''\sigma} \tag{9.2}$$

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[2] For the sake of simplicity we suppose K is finite and closed. The extension of the subsequent results to relative complexes, open complexes, and to the finite and infinite chains of locally-finite complexes will be obvious.

[3] We shall abide by the functional notation throughout. This will prove to be as convenient as the classical linear form notation. We abandon the latter since it has algebraic implications more prejudicial than suggestive in the present discussion.

By means of the isomorphisms h, we can define the boundary ∂f and coboundary δf of a q-chain f:

$$\begin{aligned} \partial f(\sigma^{q-1}) &= \sum_{\sigma^q} [\sigma^{q-1}:\sigma^q] h_{\sigma^q \sigma^{q-1}} (f(\sigma^q)), \\ \delta f(\sigma^{q+1}) &= \sum_{\sigma^q} [\sigma^q:\sigma^{q+1}] h^{-1}_{\sigma^q \sigma^{q+1}} (f(\sigma^q)). \end{aligned} \tag{9.3}$$

The sums extend over the q-cells which (a) have σ^{q-1} as a face, (b) are faces of σ^{q+1}. These new functions are $q-1$ and $(q+1)$-chains respectively. It follows from (9.1) and (9.2) that $\partial\partial f = 0$, $\delta\delta f = 0$. Therefore cycles, bounding cycles, homology and cohomology groups can be defined as usual.

A special convention for 0-cycles is necessary. We shall agree that any 0-chain is a 0-cycle. Note that the Kronecker index (i.e. the sum of the coefficients of a 0-chain) has no meaning unless the system $\{G_x\}$ is simple.[4]

Finally it is to be remarked that we obtain a completely isomorphic situation if we choose new representatives $y(\sigma)$ in each σ. The isomorphism is established by means of a path $x(\sigma)$ to $y(\sigma)$ in σ for each σ.

10. Automorphisms of chains

Let A be a group of *uniform* automorphisms or a ring of *uniform* operators of $\{G_x\}$. For any chain f and $a \in A$, define $(af)(\sigma) = af(\sigma)$ for each σ. Then af is a chain, and A appears as a group of automorphisms or a ring of operators of the group of q-chains for each q.

Since the operations of A commute with translations of the G_x along paths, it follows that the operations of A on the chains commute with ∂ and δ. Therefore A appears as an automorphism group or ring of operators of the groups of cycles, cocycles, boundaries, coboundaries, and consequently of the homology and cohomology groups.

11. Multiplication of chains

It should be noted that a cycle with local coefficients is locally a cycle in the ordinary sense; for, in any simply connected open set U (e.g. the star of a vertex), isomorphisms of the local groups can be set up with a fixed group (using paths in U) in such a way as to transform each chain with local coefficients into an ordinary chain mod $(K - U)$ so that the boundary relations are preserved. It follows that any operation on ordinary chains which is a sum of local operations can be carried over to chains with local coefficients. We have seen that this is true of the boundary operator. We shall see that this is also true of products of cocycles, products of cycles and cocycles, and intersections of cycles. The linking number like the Kronecker index is not of this category.

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[4] Classical homology with a single coefficient group G is isomorphic to homology with coefficients in the simple system of local groups determined by G.

In a comprehensive paper of Whitney on products [17], it is proved that, corresponding to cells σ_i^p , σ_j^q , σ_k^{p+q} in K, there is an integer ${}^{pq}\Gamma_k^{ij}$ such that

(Γ_1) if σ_i^p , σ_j^q are not both faces of σ_k^{p+q}, then ${}^{pq}\Gamma_k^{ij} = 0$,
(Γ_2) for all p, q, i, j, k,

$$\sum_m [\sigma_m^{p+q}:\sigma_k^{p+q+1}]^{pq}\Gamma_k^{ij} = \sum_m [\sigma_i^p:\sigma_m^{p+1}]^{p+1,q}\Gamma_k^{mj} + (-1)^p\sum_m [\sigma_j^q:\sigma_m^{q+1}]^{p,q+1}\Gamma_k^{im},$$

(Γ_3) for all q and j, $\sum_i {}^{0q}\Gamma_j^{ij} = 1$.
Although the last two conditions appear not to be local in nature, they are so by virtue of the first.

Let $\{G_x\}$ be a system of local rings with units. If f^p and f^q are p and q-chains respectively with coefficients in $\{G_x\}$, we define their *cup* product to be the $(p + q)$-chain

$$(11.1) \qquad f^p \cup f^q(\sigma_k^{p+q}) = \sum_{ij} {}^{pq}\Gamma_k^{ij}[h_k^{p+q,p}{}_{,i}(f^p(\sigma_i^p))][h_k^{p+q,q}{}_{,j}(f^q(\sigma_j^q))].$$

Here we heve abbreviated the isomorphism of the local group of σ_i^p on that of σ_k^{p+q} by $h_k^{p+q,p}{}_{,i}$. The sum extends over the faces σ_i^p , σ_j^q of σ_k^{p+q}. The relations

(P_1) $f^p \cup f^q$ is zero on any σ^{p+q} which has not both a face in f^p and a face in f^q,
(P_2) $\delta(f^p \cup f^q) = \delta f^p \cup f^q + (-1)^p f^p \cup \delta f^q$,
(P_3) $I \cup f^q = f^q$,

follow from the relations (Γ), the transitivity of the h's (9.2), and the preservation of the products in G_x under an h. In (P_3), I is the 0-cocycle which attaches to each vertex V the unit of G_V . Since h is a ring isomorphism, it preserves the unit; therefore $\delta I = 0$. It follows just as in Whitney [17] that a product is definable for the cohomology classes with the usual properties. The associative law and the commutation rule for the special products in a simplicial complex are given local proofs. Once invariance under subdivision has been established (see §16), the same laws will hold in a general complex.

For the *cap* product, we shall suppose that $\{G_x\}$, $\{H_x\}$ are paired with respect to $\{L_x\}$. Given a q-cochain f^q (coef. $\{G_x\}$) and a $(p + q)$-chain g^{p+q} (coef. $\{H_x\}$), we define their cap product to be the p-chain

$$(11.2) \qquad f^q \cap g^{p+q}(\sigma_m^p) = \sum_{jk} {}^{pq}\Gamma_k^{mj}[h_{m,j}^{p,q}(f^q(\sigma_j^q))][h_{m,k}^{p,p+q}(g^{p+q}(\sigma_k^{p+q}))]$$

with coefficients in $\{L_x\}$. The sum extends over those j, k for which σ_m^p , σ_j^q are faces of σ_k^{p+q}. Just as before, we obtain

(Q_1) $f^q \cap g^{p+q}$ is zero on any σ^p which with no σ^q of f^q is a face of a σ^{p+q} of g^{p+q},

(Q_2) $\partial(f^q \cap g^{p+q}) = (-1)^p\delta f^q \cap g^{p+q} + f^q \cap \partial g^{p+q}$.

In order to obtain an analogue of (Γ_3), we shall take the special case where $\{L_x\} = \{H_x\}$ and $\{G_x\}$ is a system of operator rings for $\{H_x\}$ (see §5). Then the 0-cocycle I is defined, and we have (Q_3) $I \cap g^p = g^p$.

Under the same assumption, it can be shown that

$$(f^q \cup f^r) \cap g^{p+q+r} \sim f^q \cap (f^r \cap g^{p+q+r}),$$

for cocycles f^q, f^r and a cycle g^{p+q+r}.

The uniqueness of the products in the following sense can be proved. Suppose ${}^{pq}\overline{\Gamma}_k^{ij}$ is another set of Γ's for which the relations (Γ) hold. Corresponding to the two sets of Γ's is the Whitney operation $\wedge$ (Theorem 8, [17]). Let ${}^{pq}\Delta_k^{ij}$ be the coefficient of σ_k^{p+1} in the product $\sigma_i^q \wedge \sigma_j^{p+q}$. Define $f^q \wedge g^{p+q}$ with equations analogous to (11.2) using Δ in place of Γ. Then the Whitney relations (R) (loc. cit.) with chains in place of cells can be proved to hold. It follows that the two sets of Γ's determine the same products among the homology and cohomology classes.

12. Duality

Let L be the group of real numbers mod 1, and let $\{L_x\}$ be the corresponding simple system in K. Let $\{G_x\}$ and $\{H_x\}$ be character systems of one another with respect to $\{L_x\}$. Since $\{L_x\}$ is simple, a 0-cycle f^0 with coefficients in $\{L_x\}$ may be regarded as a 0-cycle with coefficients in L. It possesses therefore a Kronecker index (= the sum of its coefficients) which we denote by (f^0). As usual $(f^0) = 0$ is equivalent to $f^0 \sim 0$. The scalar product of a q-cochain f^q coef. $\{G_x\}$ and a q-chain g^q coef. $\{H_x\}$ is defined to be the Kronecker index of their cap product:

$$f^q \cdot g^q = (f^q \cap g^q) \quad \text{in } L. \tag{12.1}$$

If f^q and g^q are zero except on a single σ, then, by Γ_3, $f^q \cdot g^q$ is the product $f^q(\sigma)g^q(\sigma)$. Therefore, due to the linearity, the scalar product of arbitrary f^q, g^q is the sum of products of corresponding coefficients. It follows that the groups of q-cochains and q-chains are character groups of one another.

For any q-cochain f^q and $(q+1)$-chain g^{q+1} we obtain from Q_2 that

$$\delta f^q \cdot g^{p+1} = f^q \cdot \partial g^{q+1}. \tag{12.2}$$

It follows now in the usual way (see Whitney [17]) that the q^{th} cohomology group coef. $\{G_x\}$ and the q^{th} homology group coef. $\{H_x\}$ are character groups with the scalar product as the multiplication.

13. Intersection in an orientable manifold

Let K be an orientable simplicial n-manifold and let K^* be its dual. Denote by $\mathfrak{D}\sigma$ the cell of K^* dual to the oriented simplex σ of K relative to a fixed choice of the fundamental n-cycle Z^n with integer coefficients. Let $\{G_x\}$ be a system of local groups to be used as coefficients in both K and K^*. Let the coefficient groups of σ and $\mathfrak{D}\sigma$ be the group G_x where x is their common point. Then for any chain f of K (cochain f^* of K^*), the equation

$$f^*(\mathfrak{D}\sigma) = f(\sigma) \tag{13.1}$$

defines its *dual* cochain (chain) of K^* (K) of the complementary dimension. This isomorphism between the two groups of chains has, as usual, the property

$$(\partial f)^* = (-1)^q \delta f^*, \qquad q = \text{dimension of } f. \tag{13.2}$$

It follows that *the q^{th} homology group coef. $\{G_x\}$ and the $(n-q)^{\text{th}}$ cohomology group coef. $\{G_x\}$ are isomorphic.*

In case $\{G_x\}$ is a system of local rings, we have as in §11 a multiplication defined for the cochains of K^*. The isomorphisms just established between the chains of K and cochains of K^* enable us to carry over the product in K^* into an intersection in K. We define the *intersection* of two chains f_1, f_2 of K to be the dual of the cup product of their duals:

$$f_1 \circ f_2 = (f_1^* \cup f_2^*)^*. \tag{13.3}$$

(Compare Whitney [17], p. 422, formula (19.9)). It follows that *a system of local rings in an orientable manifold determines an intersection ring of cycles isomorphic to the ring of cocycles (same coefficients) under the operation of dual.*

As is well known, the ring of integers is a ring of operators for any group G which commute with any automorphism of G. The ring of integers is therefore a uniform operator ring (§5) for any $\{G_x\}$. The equation (11.1) may be interpreted as defining the cup product of a cochain f^p with (simple) integer coefficients and a cochain f^q with local coefficients $\{G_x\}$. The relations (P) still hold, and these together with the associative law lead to the conclusion that *the cohomology ring with simple integer coefficients constitutes a ring of operators for the cohomology groups coef. $\{G_x\}$.*

The dual of this last result is that *the intersection ring of an orientable manifold with simple integer coefficients is a ring of operators for the homology groups coef. $\{G_x\}$.*

14. Intersection in a non-orientable manifold

In a non-orientable manifold K there is no n-cycle with simple integer coefficients. One cannot therefore determine the orientation of $\mathfrak{D}\sigma$ uniformly over K so that (13.2) holds. A customary device is to use integers mod 2 as coefficients so as to restore the basic n-cycle and escape orientation difficulties. The resulting duality and intersection theory is a bit weak due to the inadequacy of the coefficients. A more ingenious device has been used by de Rham [13]. We shall see that a suitable use of local coefficients permits a full development of De Rham's notion and leads to a complete and satisfying duality and intersection theory in a non-orientable manifold.

Since K is non-orientable, the elements of its fundamental group F divide into two classes according as they do or do not preserve orientation. Those which do form an invariant subgroup F^1 of index 2. Let T be the group of integers. For each integer $t \in T$ and $\alpha \in F$, let $\alpha(t)$ be $+t$ or $-t$ according as α is or is not in F^1. In this way F is a group of automorphisms of T. Let $\{T'_x\}$ be the corresponding system of local groups given by Theorem 1. We shall say that chains with coefficients in $\{T'_x\}$ have *twisted integer coefficients.*

Let G be an abelian group and let F be represented as a group of automorphisms of G. Let $\bar{G}$ be the direct sum of two copies of G (i.e. the group of pairs (g_1, g_2)). Identify G with the subgroup of elements of the form $(g, 0)$, and call G the real part of $\bar{G}$. The subgroup G' of elements of the form $(0, g)$ we call the imaginary part of $\bar{G}$. The product of (g_1, g_2) with the complex number $a + ib$ (a, b are integers) is defined by

$$(a + ib)(g_1, g_2) = (ag_1 - bg_2, ag_2 + bg_1). \tag{14.1}$$

It follows immediately that the complex integers form a ring of operators for the group $\bar{G}$, and that each element of $\bar{G}$ can be written uniquely in the form $g_1 + ig_2$ (g_1, g_2 real). If we set

$$\alpha(g_1 + ig_2) = \begin{cases} \alpha(g_1) + i\alpha(g_2) & \text{if } \alpha \in F^1, \\ \alpha(g_1) - i\alpha(g_2) & \text{if } \alpha \notin F^1, \end{cases} \tag{14.2}$$

the automorphisms of F in G are extended to $\bar{G}$. For any complex integer $a + ib$, define $\alpha(a + ib) = a \pm ib$ according as α is or is not in F^1. Let $\overline{T}$ be the ring of complex integers, and let $\{\overline{T}_x\}$ be the system of local rings corresponding to $\overline{T}$ and these automorphisms. It follows that $\{\overline{T}_x\}$ is a system of operator rings for the system $\{\bar{G}_x\}$ corresponding to $\bar{G}$ and the automorphisms (14.2) (see §5). We refer to $\{\bar{G}_x\}$ as the *complex extension of* $\{G_x\}$.

If G is a ring, and the elements of F are ring automorphisms, we define a product in $\bar{G}$ in the usual way: $(g_1, g_2)(g_1', g_2') = (g_1g_1' - g_2g_2', g_2g_1' + g_1g_2')$. The equations (14.2) define ring automorphisms of $\bar{G}$. Furthermore the elements of $\overline{T}$ associate and commute with the multiplications in $\bar{G}$. Thus $\{\bar{G}_x\}$ is a system of local rings with $\{\overline{T}_x\}$ as a system of operator rings.

We shall use $\{\bar{G}_x\}$ and $\{\overline{T}_x\}$ as coefficients for chains of K and cochains of K^*. The group of n-cycles of K coef. $\{\overline{T}_x\}$ is infinite cyclic. A generator is constructed as follows. Choose an oriented n-cell σ and let $Z^n(\sigma) = \pm i$ in $\overline{T}_\sigma$ (because of $\alpha(i) = \pm i$, the sign of i has only a local significance). If $|\sigma'|$ is any other n-cell, choose a path α of n-cells from σ to $|\sigma'|$ (successive cells having an $(n-1)$-face in common). The path α determines an orientation σ' of $|\sigma'|$ concordant with that of σ. Now define $Z^n(\sigma') = \alpha(Z^n(\sigma))$. In words: the orientation and coefficient of σ' are determined by translating along a path α both the orientation and coefficient of σ. Translating along a second path β will either produce the same orientation and the same coefficient or reverse the sign of both according as $\alpha\beta^{-1}$ is or is not in F^1. Thus the chain Z^n is independent of the paths chosen in its construction. It is a cycle since, in any simply connected domain, it is a cycle. The usual argument shows that any n-cycle of K coef. $\{\overline{T}_x\}$ is an integral multiple of Z^n. Thus the fundamental n-cycle of K has pure imaginary coefficients (or equally well, twisted integer coefficients).

If ϵ is an oriented simply-connected neighborhood in K, $Z^n(\epsilon)$ will be the coefficient $Z^n(\sigma)$ of an n-cell σ in ϵ oriented concordantly with ϵ. Corresponding to a q-cell $\sigma \in K$ and an oriented neighborhood ϵ of σ, there is in the usual way a

unique orientation of the dual cell $\mathfrak{D}_\epsilon\sigma$ in K^*. If f is any q-chain of K coef. $\{\bar{G}_x\}$, we define its dual to be the $(n-q)$-cochain f^* in K^* coef. $\{\bar{G}_x\}$ defined by

$$f^*(\mathfrak{D}_\epsilon\sigma) = -f(\sigma)Z^n(\epsilon). \tag{14.3}$$

It is to be understood that $f(\sigma)$ is in $\bar{G}_x$ and $Z^n(\epsilon)$ is in $\overline{T}_x$ where x is the point common to σ and $\mathfrak{D}_\epsilon\sigma$. Since reversing the orientation of ϵ changes the sign of both sides of (14.3), f^* is independent of the choices of the ϵ's. The dual of a chain f^* in K^* is given by

$$f^{**}(\sigma) = f^*(\mathfrak{D}_\epsilon\sigma)Z^n(\epsilon). \tag{14.4}$$

Since $Z^n(\epsilon)Z^n(\epsilon) = -1$, we have that *any chain is the dual of its dual.* The dual of Z^n is the unit 0-cocycle I. The dual of a real (imaginary) chain is imaginary (real). The formula (13.2) now follows for the dual in a non-orientable manifold; for it is a statement of local properties, and (14.3), (14.4) differ from (13.1) locally by a constant factor. As in §13, it follows that *the q^{th} homology group* coef. $\{\bar{G}_x\}$ *and the $(n-q)^{\text{th}}$ cohomology group* coef. $\{\bar{G}_x\}$ *are isomorphic.* Using (13.3) to define the intersection, we obtain a homology ring isomorphic to the cohomology ring whenever the coef. $\{\bar{G}_x\}$ form a system of local rings. In any case, the homology ring coef. $\{\overline{T}_x\}$ forms a ring of operators for the homology groups coef. $\{\bar{G}_x\}$.

Since $\bar{G}$ is the direct sum of its real and imaginary part, any chain is uniquely a sum of a real chain and an imaginary chain. The operations ∂ and δ preserve the property of being real or imaginary. Therefore the homology and cohomology groups decompose into direct sums of their real and imaginary parts. Since passing to the dual interchanges real and imaginary, we obtain the following results. *The q^{th} homology group* coef. $\{G_x\}$ (coef. $\{G'_x\}$) *is isomorphic to the $(n-q)^{\text{th}}$ cohomology group* coef. $\{G'_x\}$ (coef. $\{G_x\}$). *The homology classes* coef. $\{G'_x\}$ (*i.e. the imaginary ones*) *form a ring isomorphic to the cohomology ring* coef. $\{G_x\}$. *The intersection of two real cycles is imaginary. The intersection of a real and an imaginary cycle is real.*

It is to be noted that the results of this section apply to an orientable manifold. The absence of orientation reversing paths in no wise invalidates the constructions. We have in this way a single theory including both types of manifolds.

The classical approach to intersection is to define directly the intersection of a p-chain f of K with a q-chain g of K^* to be a chain of the subdivision of K. We may do this here as follows. If σ is a p-simplex, σ' an $(n-q)$-face of σ and ϵ an oriented neighborhood of σ, define $\sigma\circ\mathfrak{D}_\epsilon\sigma'$ relative to ϵ in the usual way. Then define the intersection chain $f\circ g$ to have on $\sigma\circ\mathfrak{D}_\epsilon\sigma'$ the value $-f(\sigma)g(\mathfrak{D}_\epsilon\sigma')Z^n(\epsilon)$.

15. The Poincaré duality

If we assume that $\{G_x\}$, $\{H_x\}$ are character systems of one another with respect to the simple system $\{L_x\}$ of mod 1 groups, we may combine the results of §12 with those of §13 and §14 to obtain: *The q^{th} homology group of the manifold K*

coef. $\{G_x\}$ (coef. $\{G'_x\}$) *and the* $(n-q)^{\text{th}}$ *homology group* coef. $\{H'_x\}$ (coef. $\{H_x\}$) *are character groups of one another; the multiplication is determined by the scalar product* $(f^* \frown g)$ *where* f *is a* q*-chain of* K *and* g *is an* $(n-q)$*-chain of* K^*.

The results simplify in the orientable case if we note that $\{G_x\}$ and $\{G'_x\}$ are isomorphic, as also are $\{H_x\}$ and $\{G'_x\}$. In the non-orientable case, the results simplify if we note that $\{H'_x\}$ ($\{H_x\}$) is the character system of $\{G_x\}$ ($\{G'_x\}$) with respect to the system $\{L'_x\}$ of twisted mod 1 groups associated with the simple system $\{L_x\}$.

Before leaving the subject of duality, let us observe that the notion of local coefficients has nothing to add to the duality theorem of Alexander. A vital step in the argument of Alexander is the following: *A cycle in the closed set* R *in the* n*-sphere* S^n *is a cycle in* S^n *and is therefore the boundary of a chain in* S^n. Such a statement is valid only if the system of local coefficients used in R is part of a system in S^n; as S^n is simply-connected ($n > 1$), the system of local coefficients must be simple.

16. Chain mapping and subdivision

A chain transformation is a homomorphism of the groups of chains of one complex on those of another which commutes with the boundary or coboundary operator or interchanges them (as in the case of the dual). This definition has meaning of course when local coefficients are used. All that we need to determine here is that chain transformation "S" with local coefficients exist in the usual circumstances.

Let a cell mapping $K' \to K$ be given which preserves the relation of incidence. Let $\{G_x\}$ be a system of local coefficients in K, and let $\{G'_y\}$ be the induced system in K' (see §7). If $\sigma' \to \sigma$, there is an attached isomorphism $G'_{\sigma'} \to G_\sigma$. A chain of K' which is zero except on a single σ' we call an *elementary chain*. Let f' be an elementary chain and $f'(\sigma') \neq 0$. If the image σ of σ' has a lower dimension, we define the image f of f' to be zero. If σ, σ' have the same dimension, the image f of f' is zero except on σ, and $f(\sigma)$ is the image of $f'(\sigma')$ under the isomorphism $G'_{\sigma'} \to G_\sigma$. An arbitrary chain of K' is uniquely a sum of elementary chains. Its image is defined as the sum of the images of its elementary parts. The resulting chain mapping we denote by τ. The inverse cochain mapping τ' attaches to an elementary chain of K the sum of the elementary chains of K' mapped into it by τ; we then extend τ' preserving linearity. That τ (τ') commutes with ∂ (δ) is proved by first establishing it in the usual way for elementary chains (of course (7.1) is used), and then applying the linearity of ∂ (δ).

It is necessary to use in K' the induced system $\{G'_y\}$ in order that τ, τ' shall exist in all dimensions. This is seen as follows. Let V' be a vertex of K' and V its image. Assuming τ, τ' defined for the elementary chains of V, V^1, we arrive at an isomorphism $G'_{V'} \to G_V$. Therefore $\{G'_y\}$ is the system induced by $\{G_x\}$ over the 0-dimensional part of K'. Suppose this is known for the q-dimensional part of K'. Any closed $(q+1)$-cell is simply-connected, there is therefore just one system of local groups defined over it which agrees with a given

system on its boundary, and that one is simple. We conclude that the given system and the induced system agree on each closed $(q+1)$-cell, and finally over the whole of K'.

If K' is a subdivision of the simplicial complex K, we then have two systems in K': the given system $\{G_x\}$ for K, and the system $\{G'_x\}$ induced by the map $K' \to K$ defined by mapping each vertex of K' into a vertex of the simplex of K containing it. The two systems are isomorphic. The isomorphism is set up by using the line segments which join each point to its image point. The proof of the invariance under subdivision of the groups of K and their multiplications may now be completed in the standard way (see for example [17]).

17. Continuous cycles

Let $\{G_x\}$ be a system of local groups in a space R. A *continuous chain* in R is a collection composed of a complex K, a continuous map ϕ of K in R, and a chain Z in K with local coefficients in the system $\{G'_y\}$ induced by ϕ and $\{G_x\}$. If Z is a cycle, the collection (K, ϕ, Z) is called a *continuous cycle*. The boundary of (K, ϕ, Z) is $(K, \phi, \partial Z)$. Two continuous chains (K_i, ϕ_i, Z_i) $(i = 1, 2)$ are *added* by forming the abstract sum $K_1 + K_2$, defining $\phi = \phi_i$ on K_i, and adding Z_1 to Z_2. Two continuous cycles are *homologous* if there exists a chain (K, ϕ, Z) such that $K \supset K_1$ and K_2, $\phi = \phi_i$ on K_i, and $\partial Z = Z_1 - Z_2$. The cycles of a fixed dimension divide up into homology classes. Two classes are *added* by adding representative elements. In this way we define the homology groups of R based on continuous cycles with local coefficients $\{G_x\}$. That they are topological invariants of R and the system $\{G_x\}$ is an immediate consequence of the definition.

If R is the space of a complex, these groups of R are isomorphic to those of K with the same local coefficients. The identity map ϕ_1 of K attaches to a chain Z of K the continuous chain (K, ϕ_1, Z) of R. This chain mapping commutes with ∂, and therefore induces homomorphisms of the groups of K into those of R. That these are isomorphisms follows from the lemma: If (K', ϕ, Z') is a chain with boundary of the form (K, ϕ_1, Z), then there is a chain Z_1 of K such that $\partial Z_1 = Z$, and the difference $(K', \phi, Z') + (K, \phi_1, -Z_1)$ bounds a continuous chain in R. The lemma is proved in the usual way by using the simplicial approximation theorem to construct a map of the product complex $K' \times I$ $(I = (0, 1))$ into R. The needed chain is found in $K' \times I$ with local coefficients in the induced system.

18. Čech cycles

The only difficulty in the way of extending local coefficients to Čech cycles is that of constructing a system of local groups in the nerve K of a finite open covering when such a system is given in R. It is clear that the former must be chosen so as to induce the given system in R under the natural map $R \to K$. If R is sufficiently complicated, it is possible to construct in R a local system

which is not induced by a local system in any nerve.[5] Therefore we are forced to restrict ourselves to a system $\{G_x\}$ induced in R by a system $\{G_x^0\}$ in a fixed nerve K^0. We then admit only those coverings which are refinements of K^0, and we use in them the local groups induced by their natural projections into K^0. With this modification, the definitions of the Čech homology and cohomology groups and their multiplications proceed as before.

If R is the space of a complex K, a local system in R is one in $K = K^0$. Using invariance under subdivision, one proves in the customary way (see [15], §9) that the groups of K and the Čech groups of R are isomorphic, and the isomorphisms preserve the multiplications. We are thus led to a proof that the homology theory (coef. $\{G_x\}$) of a complex K is a topological invariant of the space K with the local groups $\{G_x\}$.

19. Überdeckung

In a complex K choose a reference point o (preferably a vertex), and for each cell σ let α_σ be a path in K from o to a point $x(\sigma)$ in σ. If $\sigma' < \sigma$, let $\alpha_{\sigma\sigma'}$ be a path in the closure of σ from $x(\sigma)$ to $x(\sigma')$. The closed path $\alpha_\sigma\alpha_{\sigma\sigma'}\alpha_{\sigma'}^{-1}$ is abbreviated by $\gamma_{\sigma\sigma'}$. As elements of the fundamental group F of K (origin o), the γ's have the property

$$\gamma_{\sigma\sigma'}\gamma_{\sigma'\sigma''} = \gamma_{\sigma\sigma''} \qquad \text{for } \sigma'' < \sigma' < \sigma. \tag{19.1}$$

By means of the α's, we map isomorphically the chains of K with local coef. $\{G_x\}$ into ordinary chains with coefficients in G_o as follows. The transform $\tilde{f}$ of the chain f with local coefficients is defined by $\tilde{f}(\sigma) = \alpha_\sigma^{-1}(f(\sigma))$. By means of this isomorphism, we define operators $\bar{\partial}$, $\bar{\delta}$ for chains $\tilde{f}$ by: $\bar{\partial}\tilde{f} = \widetilde{\partial f}$, $\bar{\delta}\tilde{f} = \widetilde{\delta f}$. From (9.3) we obtain

$$\begin{aligned} \bar{\partial}\tilde{f}(\sigma') &= \textstyle\sum_\sigma [\sigma' : \sigma]\gamma_{\sigma\sigma'}(\tilde{f}(\sigma)), && \sigma' < \sigma, \\ \bar{\delta}\tilde{f}(\sigma'') &= \textstyle\sum_\sigma [\sigma : \sigma'']\gamma_{\sigma''\sigma}^{-1}(\tilde{f}(\sigma)), && \sigma < \sigma''. \end{aligned} \tag{19.2}$$

Thus the system of ordinary chains (coef. G_o) with the special operators $\bar{\partial}$, $\bar{\delta}$ are isomorphic to the system of chains with local coef. $\{G_x\}$ with the ordinary operators ∂, δ. They determine therefore isomorphic homology groups. It is a corollary that the homology groups determined by $\bar{\partial}$, $\bar{\delta}$ are independent of the choices of the α's.

The system $\tilde{f}$, $\bar{\partial}$, $\bar{\delta}$ just described is called an *Überdeckung* of K by Reidemeister [10; 11]. An advantage of this approach is that it lends itself more readily to a computation of the homology groups. One may attempt to simplify the incidence matrices $[\sigma' : \sigma]\gamma_{\sigma\sigma'}$, with elements in the group ring of F, by the usual methods of transforming bases and consolidation (see W. Franz [2]).

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[5] This is the case if R is not locally simply-connected, and if $\{G_x\}$ is the system $\{G_x^L\}$ of local group rings of §8.

20. Zero and 1-dimensional groups

If the fundamental group F of K and the operations of F in G_o are given, then one may compute the 0 and 1-dimensional homology groups without further knowledge of K. Choose a basis $\alpha_1, \cdots, \alpha_h$ of F and a basis $r_1, \cdots, r_s$ for the relations in F (each r is a product of α's representing the unit). Construct a 2-complex K' consisting of one vertex o', one edge for each α_i (likewise denoted by α_i) with both end-points at o', and, for each r_i, a 2-cell E_i whose boundary is the product r_i of the α's. Clearly F is also the fundamental group of K'. The operations of F in G_o determine local coefficients in K' leading to 0 and 1-dimensional homology and cohomology groups which we shall prove isomorphic to those of K. By the duality of §12, it suffices to prove this for the homology groups.

Define a map ϕ of K' into K so that $\phi(o') = o$, $\phi(\alpha_i)$ represents α_i, and ϕ is continuous. It is readily seen that a map ψ of a complex K'' in K is homotopic to a map ψ' which, on the 1-dimensional part K_1'' of K'', can be expressed as a product $\phi\psi''$ of ϕ and a map ψ'' of K_1'' in K' which maps each vertex into o' and each edge into a product of α's. Thus every continuous 0 and 1-cycle is homologous to one of the form (K', ϕ, Z) (see §17). It is a further consequence that the 0-cycle Z bounds in K' if (K', ϕ, Z) bounds in K. This shows that the 0^{th} homology groups of K and K' are isomorphic. To prove the same for the 1-dimensional groups, we must show that a homology relation in K of the form $\partial(E, \psi', f(E) = g) = (K', \phi, Z)$, where E is a 2-cell and $\psi'(\partial E)$ is a product r of the α's, is a consequence of the relations in K'. Let E be regarded as a hemisphere of a 2-sphere S^2 and let E' be the other hemisphere. The map ψ'' of the equator in K' extends to a continuous map ψ'' of E' in K'; for the product r is expressible in terms of the r_i. The map ψ' of E and $\phi\psi''$ of E' define a map ψ' of S^2 in K, and thereby a 2-cycle $(S^2, \psi', f(E) = f(-E) = g)$. Thus $\partial(E', \psi', f(E') = g) = (K', \phi, Z)$, and ψ' factors into $\phi\psi''$.

Using the above results we may describe the 0-dimensional groups quite easily. Let G_o' be the subgroup of elements of G_o which are fixed under every automorphism $\alpha \in F$. *The 0^{th} cohomology group of K is isomorphic to the group G_o'.* Let G_o'' be the subgroup of elements of G_o expressible in the form $\sum_i(\alpha_i(g_i) - g_i)$ where $\alpha_i \in F$, $g_i \in G_o$. *The 0^{th} homology group of K is isomorphic to the difference group $G_o - G_o''$.*

It is worth noting that a continuous image of an n-sphere ($n > 1$) in K determines a group of spherical n-cycles for any local coefficients; for the sphere is simply connected. However these cycles may or may not bound according to the structure of the system of local groups. Consider, as an example, the projective plane P^2 and the double covering of it by the 2-sphere S^2. With twisted integer coefficients (§14), the 2-cycles on P^2 form an infinite cyclic group and are nonbounding. The even multiples of the generator are images of the 2-cycles on S^2 with integer coefficients. Yet with simple integer coefficients in P^2, the mage of every 2-cycle on S^2 is bounding (algebraically zero).

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Bibliography

1. P. Alexandroff-H. Hopf, *Topologie*, Berlin, 1936.
2. W. Franz, *Überdeckungen topologischer Komplexe mit hyperkomplexen Systemen*, J. reine angew. Math., 173 (1935), 174–184.
3. ———, *Über die Torsion einer Überdeckung*, ibid., 173 (1935), 245–254.
4. ———, *Torsionideale, Torsionklassen und Torsion*, ibid., 176 (1936), 113–124.
5. A. Komatu, *Über die Dualitätssätze der Überdeckungen*, Jap. Journ. Math., 13 (1937), 493–500.
6. ———, *Über die Überdeckungen von Zellenräumen*, Proc. Imp. Acad. Tokyo, 14 (1938), 340–341; 15 (1939), 4–7; 16 (1940), 55–58.
7. S. Lefschetz, *Topology*, Colloquium lectures, New York, 1930.
8. K. Reidemeister, *Homotopiegruppen von Komplexen*, Hamburger Abh., 10 (1934).
9. ———, *Homotopieringe und Linsenräume*, ibid., 11 (1935).
10. ———, *Überdeckungen von Komplexen*, J. reine angew. Math., 173 (1935), 164–173.
11. ———, *Topologie der Polyeder*, Leipzig, 1938.
12. ———, *Durchschnitt und Schnitt von Homotopieketten*, Monats. für Math., 48 (1939), 226–239.
13. G. de Rham, *Sur la théorie des intersections et les integrales multiples*, Comment. Math. Helv., 4 (1932), 151–157.
14. ———, *Sur les complexes avec automorphismes*, ibid., 12 (1940), 191–211.
15. N. Steenrod, *Universal homology groups*, Amer. Journ. Math., 58 (1936), 661–701.
16. ———, *Topological methods for the construction of tensor functions*, Annals of Math., 43 (1942).
17. H. Whitney, *On products in a complex*, Annals of Math., 39 (1938), 397–432.
18. ———, *On the theory of sphere-bundles*, Proc. Nat. Acad. Sci., 26 (1940), 148–153.
19. ———, *On the topology of differentiable manifolds*, Lectures in Topology, Univ. of Michigan, 1940.

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NON-COMMUTATIVE CHAINS, II

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Non-commutative chains of dimension two are here set up for a finite abstract system and by means of them an abelian "homology group" μ_2 is found in the spirit of the author's earlier definition of the Poincaré group as a homology group[2]. The group μ_2 is subdivision invariant and, furthermore, each finite simplicial complex determines a family of systems all having the same μ_2. At the end of this paper μ_2 is computed for some simplicial complexes including the lens spaces. For the lens spaces μ_2 is always the identity.

A recent paper by Hopf[3] has obvious points of contact with this one: each 2-syllable of this paper has as boundary one of the "loop paths" defining the "homotopy boundary" of the 2-cell in question in the sense of Hopf; if the C of (1.2) below is a cycle, the chain $E_{i_1}^{x_1} + \cdots + E_{i_s}^{x_s}$ is a Hopf spherical cycle.

1. The abstract *system* S consists of "cells", each having associated with it an integer called its dimension, and an operator F (meaning boundary) whose domain is the set of cells of S and whose range is a subset of the "chains" of S. The cells comprize the neutral cell, 1, and n-dimensional cells (n-cells, E_i^n) in finite number for $n = 0, 1, 2, 3$. It is convenient to suppose that 1 is an n-cell for each n.

By an n-chain when $n = 0, 1$ (but not 2, 3) is meant a class of "words" in the sense of the theory of non-abelian groups with a finite number of generators, subject to the relation $EE^{-1} = 1$, the letters of a word being n-cells or their inverses, for instance:

$$C = (E_{i_1}^n)^{x_1}(E_{i_2}^n)^{x_2} \cdots (E_{i_s}^n)^{x_s}, \qquad x_j = \pm 1. \tag{1.1}$$

The inverse of a chain and the product of two chains are then defined in the obvious way.[2] For the cell 1 and for zero and one-cells E, FE is defined as follows:

$$F1 = 1$$

$$FE_i^0 = 1 \text{ for zero-cells } E_i^0$$

$$FE_i^1 = E_{i_1}^0(E_{i_2}^0)^{-1} \text{ for one-cells } E^1 \text{ where } i_1, i_2 \text{ depend on } i.$$

As in N.C., $E_{i_1}^0$ and $(E_{i_2}^0)^{-1}$ are called the *outset* and *finish* of E_i^1. A word (1.1) has property ϕ if it $\equiv 1$ or if the finish of $(E_{i_j}^1)^{x_j}$ for $j = 1, 2, \cdots, s-1$ is the

[1] Presented to the Society, December 1941.

[2] W. W. Flexner, Non-commutative Chains and the Poincaré Group, Duke Journal 8 (1941) pp. 497–505. Here referred to as N.C.

[3] Hopf, Heinz, *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv. 14 (1942) pp. 257–309.

inverse of the outset of $(E^1_{i_{j+1}})^{x_{j+1}}$. The outset of $(E^1_{i_1})^{x_1}$ is called the outset of C^1 and the finish of $(E^1_{i_s})^{x_s}$ is called the finish of C^1. The system S is now subjected to condition Γ of N.C. p. 498 making S connected. If C^1 has property ϕ outset E^0 and finish $(E^0)^{-1}$, C is said to be a *one-cycle on* E^0. The one-cycles on E^0 form a subgroup $\mathfrak{Z}_1$ of the group of one-chains.

To each 2-cell E^2_i a one-cycle FE^2_i on some fixed zero-cell, call it ω, is now associated. (This FE^2_i bears to the boundary defined in N.C., p. 498, call it $\bar{F}E^2_i$, the relation $FE^2_i = a\bar{F}E^2_i a^{-1}$, where a is a 1-chain having property ϕ and outset ω). If a_i is a one-cycle on ω and E_i is a 2-cell or the inverse of a 2-cell the following symbolic combination Ω_i of a_i and E_i is called a 2-*syllable*:

$$\Omega_i = a_i E_i a_i^{-1}.$$

The inverse and boundary of Ω_i are defined to be

$$\Omega_i^{-1} = a_i E_i^{-1} a_i^{-1}$$
$$F\Omega_i = a_i (FE_i) a_i^{-1}.$$

If C and D are "words" composed of 2-syllables, for instance,

$$C = (a_1 E_{i_1}^{x_1} a_1^{-1})(a_2 E_{i_2}^{x_2} a_2^{-1}) \cdots (a_s E_{i_s}^{x_s} a_s^{-1}), \qquad x_k = \pm 1,$$
(1.2)
$$D = (b_1 E_{j_1}^{y_1} b_1^{-1})(b_2 E_{j_2}^{y_2} b_2^{-1}) \cdots (b_t E_{j_t}^{y_t} b_t^{-1}), \qquad y_k = \pm 1,$$

let $CD = (a_1 E_{i_1}^{x_1} a_1^{-1}) \cdots (a_s E_{i_s}^{x_s} a_s^{-1})(b_1 E_{j_1}^{y_1} b_1^{-1}) \cdots (b_t E_{j_t}^{y_t} b_t^{-1})$ and

$$FC = (a_1 FE_{i_1}^{x_1} a_1^{-1}) \cdots (a_s FE_{i_s}^{x_s} a_s^{-1}).$$

Then

$$F(CD) = (FC)(FD).$$

If subjected to the following relations between the syllables these words form the group $\mathfrak{C}_2$ of two-chains:

(r.1) $$\Omega_i \Omega_i^{-1} = 1$$

(r.2) $$\Omega_i \Omega_j \Omega_i^{-1} = (F\Omega_i)\Omega_j (F\Omega_i)^{-1}$$

(r.3) $$a_i 1 a_i^{-1} = 1, \quad a_i \in \mathfrak{Z}_1.$$

DEFINITION 1.3. *If b is a one-cycle on ω and C is defined by* (1.2), *then*

(1.4) $$b\,C\,b^{-1} = (ba_1 E_{i_1}^{x_1} a_1^{-1} b^{-1}) \cdots (ba_s E_{i_s}^{x_s} a_s^{-1} b^{-1}).$$

THEOREM 1.5. *If C is a 2-chain, b is a 1-cycle on ω and $C = 1$, then $b\,C\,b^{-1} = 1$.*

PROOF. The relations r which are used on the r.h.s. of (1.2) to reduce it to 1 can be used similarly on the r.h.s. of (1.4) and will reduce $b\,C\,b^{-1}$ to 1.

DEFINITION 1.6. *That two words C and D are identical as words will be written $C \equiv D$, that they define the same element of $\mathfrak{C}_2$ will be written $C = D$.*

THEOREM 1.7. *If $C = D$ then $FC = FD$.*

PROOF. The operator F applied to the l.h.s. of the relations r gives the same result as when applied to the r.h.s.

This shows that FC depends only on the element of $\mathcal{C}_2$ defined by C. Those elements $C \epsilon \mathcal{C}_2$ for which $FC = 1$ form a subgroup $\mathcal{Z}_2$ of $\mathcal{C}_2$ called the group of two-cycles of S on ω.

THEOREM 1.8. *Each $C \epsilon \mathcal{Z}_2$ commutes with every $D \epsilon \mathcal{C}_2$.*

PROOF. By (r.2) $C\,D\,C^{-1}D^{-1} = (FC)\,D\,(FC)^{-1}D^{-1}$

$= D\,D^{-1}$ since C is a cycle

$= 1$ by (r.1).

COROLLARY 1.9. *The group $\mathcal{Z}_2$ is abelian.*

For each 3-cell $\mathfrak{S}_i$ of S let $F\mathfrak{S}_i$ be a unique element of $\mathcal{Z}_2$. The elements C of $\mathcal{Z}_2$ which can be defined by words all of whose syllables are of the form

$$a(F\mathfrak{S}_i)^q a^{-1}, \qquad q = \pm 1, \qquad a \epsilon \mathcal{Z}_1$$

make up a subgroup, $\mathcal{F}_2$, of $\mathcal{Z}_2$. The factor group $\mu_2 = \mathcal{Z}_2/\mathcal{F}_2$ is called the *second homology group of non-commutative chains of S on ω.*

2. If S is a system then $\bar{S}$ is a system and is called an *elementary subdivision of* S if $\bar{S}$ is related to S as follows. The elements of $\bar{S}$ are obtained from those of S by replacing a single p-cell of S by two p-cells Y and Z and a $(p-1)$-cell X, $p = 1, 2, 3$. If E_1 is a cell of S such that $E_1 \neq E$ and FE_1 does not contain E, then FE_1 in $\bar{S}$ is the same as FE_1 in S. If in S

(2.1) $FE = GL$, G, L $(p-1)$-chains for $p > 1$; G a vertex and L the inverse of a vertex for $p = 1$, then in $\bar{S}$

(2.2) $FX = FL$

(2.3) $FY = GX$

(2.4) $FZ = X^{-1}L$.

If FE_1 in S contains E, FE_1 in $\bar{S}$ is the same as FE_1 in S except that E is replaced by YZ.[4]

If S' is obtained from S by a finite sequence of elementary subdivisions, S' is called a *subdivision* of S.

THEOREM 2.5. *If $\bar{S}$ is an elementary subdivision of S, $\bar{S}$ and S have the same groups μ_2.*

COROLLARY 2.6. *If S' is a subdivision of S, S' and S have the same groups μ_2.*

In future all cycles will be "on ω" unless exception is specifically made and the following notational conventions will be adhered to:

Script capitals, except $\mathcal{C}$, $\mathcal{Z}$ and $\mathcal{F}$, for 3-cells;
the printed capitals E, J, X, Y, Z, for 2-cells and their inverses;
other printed capitals, except F, for 2-chains;
small latin letters for one-chains;
small greek letters, except π, τ and μ, for zero-cells and their inverses.

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[4] For $p = 2$ this should be compared with N.C. p. 499, II. If in N.C. substitutions are made as shown below, the two definitions correspond: $FE \to C_2 FE\, C_2^{-1}$, $FE' \to C_2 FE' C_2^{-1}$, $FE'' \to FE''$. This shows that the distinction is a notational one.

3. PROOF OF THEOREM 2.5. The groups of $\bar{S}$ will be distinguished by a bar. The proof that $\mu_2 \approx \bar{\mu}_2$ is then made in the three cases: $p = 1$, $p = 2$, $p = 3$. When $p = 1$ the one-cell e is replaced by the one-cells y and z and the vertex ξ. Formulas (2.1)–(2.4) become

$$Fe = \gamma\lambda, \qquad F\xi = F\lambda, \qquad Fy = \gamma\xi, \qquad Fz = \xi^{-1}\lambda.$$

If E is a 2-cell, FE in $\bar{S}$ is obtained by replacing e in FE of S by yz. Each 2-syllable of $\bar{\mathcal{C}}_2$ is of the form $a\,Ja^{-1}$, $J^{\pm 1}$ a 2-cell of $\bar{S}$, $a \in \bar{\mathcal{Z}}_1$. If a contains y or z non-trivially it contains both together as $(yz)^{\pm 1}$ for otherwise a cannot be a one-cycle. Hence the 2-chains of $\bar{S}$ become 2-chains of S when yz is everywhere replaced by e. Thus in this case there is an isomorphism $I\bar{\mathcal{Z}}_2 = \mathcal{Z}_2$ such that $I\bar{\mathcal{F}}_2 = \mathcal{F}_2$ and so $\mu_2 \approx \bar{\mu}_2$.

4. $p = 2$. E is replaced by Y, Z and x where if

$$FE = gm, \qquad \text{then} \tag{4.1}$$

$$Fx = Fm \tag{4.2}$$

$$FY = gx \tag{4.3}$$

$$FZ = x^{-1}m. \tag{4.4}$$

Make a change of basis for the 2-chains of $\bar{\mathcal{C}}_2$ such that

$$Y \to \bar{E}Z^{-1} \tag{4.5}$$

$$J \to J \qquad \text{if } J \neq Y \text{ is a 2-cell of } \bar{S},$$

where $\bar{E}$ is a new symbol playing the role of a 2-cell of $\bar{S}$ and such that

$$F\bar{E} = FE.$$

LEMMA 4.6. *If $\bar{C} \in \bar{\mathcal{Z}}_2$ and Y has been eliminated from $\bar{C}$ by* (4.5), *there is a $\bar{D}$ containing neither x nor Z and such that $\bar{D} = \bar{C}$.*

PROOF 4.7. If the word $\bar{C}$ has a syllable $M = axbJb^{-1}x^{-1}a^{-1}$, where J is a 2-cell of $\bar{S}$, (4.4) gives that $x = m(FZ)^{-1}$. Hence

$$\begin{aligned} M &= amFZ^{-1}m^{-1}a^{-1}(ambJb^{-1}m^{-1}a^{-1})\; amFZm^{-1}a^{-1} \\ &= amZ^{-1}m^{-1}a^{-1}(ambJb^{-1}m^{-1}a^{-1})amZm^{-1}a^{-1} \qquad \text{by (r.2).} \end{aligned} \tag{4.8}$$

So $\bar{C}$ has equals containing no x. Among these there is at least one word having the fewest syllables containing Z. Let $\bar{D}$ be such a word and suppose s is the number of its syllables containing Z. If $s = 0$, lemma 4.6 is proved. If there is a syllable aZ^qa^{-1} in the word $\bar{D}$ ($q = \pm 1$), there must be another syllable $b\,Z^{-q}b^{-1}$ the x of whose boundary cancels the x in the boundary of $a\,Z\,a^{-1}$ in $F\bar{D} = 1$. As there is no loss in assuming $q = 1$, $\bar{D}$ can be written:

$$\bar{D} = U(a\,Z\,a^{-1})V(b\,Z^{-1}b^{-1})W \qquad \text{where } U, V, W \text{ are elements of } \bar{\mathcal{C}}_2. \tag{4.9}$$

Let $FU = u$, $FV = v$, $FW = w$. Then

$$F\bar{D} = u(a\,x^{-1}m\,a^{-1})v(b\,m^{-1}x\,b^{-1})w = 1,$$

and $a^{-1}v\,b = 1$ so $a = v\,b$. Therefore

$$(4.10)\qquad \begin{cases} \bar{D} = U(v\,b\,Z\,b^{-1}v^{-1})V(b\,Z^{-1}b^{-1})W \\ \quad = UV(b\,Z\,b^{-1})V^{-1}V(b\,Z^{-1}b^{-1})W \\ \quad = UVW, \end{cases}$$

which contradicts the hypothesis that $\bar{D}$ in its original form had s a minimum. Hence $s = 0$ and 4.6 is proved.

If $\bar{C} \in \bar{\mathfrak{Z}}_2$ and $\bar{C}$ as a word is free of x and Z, let $\tau\bar{C}$ be the chain of $\mathfrak{Z}_2$ obtained by replacing $\bar{E}$ in $\bar{C}$ by E in $\tau\bar{C}$. To see that $\tau\bar{C}$ depends only on the *element* $\bar{C}$ and not on the *word* $\bar{C}$ chosen to represent that element, observe that all steps (r.1), (r.2) and (r.3) which can be made for $\bar{C}$ can be reproduced for $\tau\bar{C}$ except those involving x and Z. But (4.8) and (4.9), (4.10) show that lemma 4.6 has the effect on the original word $\bar{C}$ of replacing x by m and then replacing Z by 1. Hence if m and 1 are used in $\tau\bar{C}$ where x and Z were used in $\bar{C}$ and if $F1$ be written $m^{-1}m$, the transformations of the word $\bar{C}$ can be duplicated throughout the word $\tau\bar{C}$.

Now if $C \in \mathfrak{Z}_2$, replacing E by $YZ = \bar{E}$ will yield a chain $\bar{C} \in \bar{\mathfrak{Z}}_2$ such that $\tau\bar{C} = C$, so $\tau\bar{\mathfrak{Z}}_2 = \mathfrak{Z}_2$. It is immediate that τ is a homomorphism. If $\mathcal{H}$ is a 3-cell then $F\mathcal{H}$ in $\bar{S}$ is by definition $F\mathcal{H}$ in S with E replaced by $YZ = \bar{E}$. So $\tau F\mathcal{H} = F\mathcal{H}$ and $\tau^{-1}F\mathcal{H} = F\mathcal{H}$, whence $\tau^{-1}\mathfrak{F}_2 = \bar{\mathfrak{F}}_2$ and so[5] $\mu_2 \approx \bar{\mu}_2$ when $p = 2$.

5. $p = 3$. $\mathfrak{S}$ is replaced by $\mathfrak{Y}$, $\mathcal{J}$ and X.

$$(5.1)\qquad F\mathfrak{S} = GL$$

$$(5.2)\qquad FX = FL$$

$$(5.3)\qquad F\mathfrak{Y} = GX$$

$$(5.4)\qquad F\mathcal{J} = X^{-1}L.$$

If $\bar{C} \in \bar{\mathfrak{Z}}_2$ obtain $\tau\bar{C}$ by replacing X everywhere by L. By (5.2), $\tau\bar{C}$ is a unique element of $\mathfrak{Z}_2$. If $C \in \mathfrak{Z}_2$, then $C \in \bar{\mathfrak{Z}}_2$ and $\tau C = C$ so τ is a homomorphism such that $\tau\bar{\mathfrak{Z}}_2 = \mathfrak{Z}_2$. Since $\mathcal{H} \neq \mathfrak{Y}$, $\mathcal{J}$ a 3-cell of $\bar{S}$ implies $F\mathcal{H}$ is the same in S and $\bar{S}$, $\tau(F\mathcal{H}) = F\mathcal{H}$. Also

$$\tau F\mathfrak{Y} = \tau(GX) = GL = F\mathfrak{S}$$

$$\tau F\mathcal{J} = \tau(X^{-1}L) = 1 = F1,$$

so $\bar{C} \in \bar{\mathfrak{F}}_2$ implies $\tau\bar{C} \in \mathfrak{F}_2$, whence $\tau\bar{\mathfrak{F}}_2 \subset \mathfrak{F}_2$. It remains to show that $\tau^{-1}\mathfrak{F}_2 \subset \bar{\mathfrak{F}}_2$ from which $\tau^{-1}\mathfrak{F}_2 = \bar{\mathfrak{F}}_2$ and $\mu_2 \approx \bar{\mu}_2$ will follow.

[5] B. L. van der Waerden, *Moderne Algebra*, Berlin 1930, vol. 1, p. 136.

Adopt the following notation for 2-syllables of $\bar{S}$.

$\Omega_i = b_i J_i b_i^{-1}$, where $J_i \neq X$, X^{-1} is a 2-cell of $\bar{S}$ or its inverse,

$\Lambda_i = a_i X^{q_i} a_i^{-1}$, where $q_i = \pm 1$,

$\Gamma_i = a_i L^{q_i} a_i^{-1}$. Then if

$$\bar{C} = \Omega_0 \Lambda_1 \Omega_1 \cdots \Lambda_s \Omega_s \tag{5.5}$$

$$\tau \bar{C} = \Omega_0 \Gamma_1 \Omega_1 \cdots \Gamma_s \Omega_s . \tag{5.6}$$

Take $C \in \mathcal{F}_2$ and $\bar{C} \in \mathcal{Z}_2$ such that $\tau \bar{C} = C$. Then C can be written in the form of the r.h.s. of (5.6) and $\bar{C}$ is given by (5.5). But (5.4) yields $X = L(F\mathcal{J})^{-1}$ so

$$\bar{C} = \Omega_0 [a_1 (LF\mathcal{J}^{-1})^{q_1} a_1^{-1}] \Omega_1 \cdots [a_s (LF\mathcal{J}^{-1})^{q_s} a_s^{-1}] \Omega_s . \tag{5.7}$$

If (5.7) be further expanded by definition 1.3, $\bar{C}$ becomes a product of syllables of the three kinds: Ω_i, Γ_i, $a_i F \mathcal{J}^{-q_i} a_i^{-1} = \Delta_i$. But N.C. p. 501 formula (3.3) now applies if

Δ_i be represented by the b_i of N.C.,

Ω_i be represented by a_i for some i,

Γ_i be represented by a_i for the remaining i.

Then (N.C.3.5) yields $C = g'$, and h_i will be a product of terms Ω_k and Γ_j. But with h_i occurs h_i^{-1} so by (r.2) each h_i may be replaced by a product of terms $F\Omega_k$ and $F\Gamma_j$. Hence each $h_i b_i h_i^{-1}$ of (N.C.3.4) is $h_i \Delta_m h_i^{-1} \in \bar{\mathcal{F}}_2$ so

$$\bar{C} = D\,C \qquad \text{with } D \in \bar{\mathcal{F}}_2 . \tag{5.8}$$

Since by hypothesis $C \in \mathcal{F}_2$, C is a product of terms $c\mathcal{H}c^{-1}$ where $\mathcal{H}$ is a 3-cell or its inverse. If $\mathcal{H} \neq \mathfrak{S}$, $F\mathcal{H}$ is the same in S and $\bar{S}$; if $\mathcal{H} = \mathfrak{S}$, $F\mathcal{H} = F(\mathcal{Y}\mathcal{J})$ in $\bar{S}$, so in all cases $C \in \bar{\mathcal{F}}_2$. Hence, by (5.8), $\bar{C} \in \bar{\mathcal{F}}_2$ and $\tau^{-1}\mathcal{F}_2 \in \bar{\mathcal{F}}_2$, which completes the proof of theorem 2.5 and its corollary, 2.6.

6. Now suppose K is a finite simplicial connected 3-complex. It will be shown in this section that a system S can be associated with K, not in a unique manner, but so that the group μ_2 of S is uniquely determined by K. From 2.6 it then follows that μ_2 is unchanged by subdivision of K. Therefore for π_1 (see N.C.) and μ_2, S bears the same relation to K that an "abstract complex" bears to K for the ordinary homology groups.

Let the p-cells of S be the p-cells of K for $p = 0, 1, 2, 3$. Take ω to be some vertex of K. If a is a 1-cell, set $Fa = \beta\gamma^{-1}$ where β, γ are the end points of a in either order. If E is a 2-cell, let c, d, e each be a one-cell on the boundary of E (or the inverse of such a one-cell) such that cde is one of the six cyclic orders of the one-cells around E that has property ϕ. Let b be a one-chain with property ϕ outset ω and finish equal to the inverse of the outset of c. Then

$$FE = b(cde)b^{-1}.$$

If $\mathfrak{E}$ is a 3-cell, let

$$F\mathfrak{E} = (b_1 J_1^{q_1} b_1^{-1}) \cdots (b_4 J_4^{q_4} b_4^{-1}), \qquad q_i = \pm 1,$$

where the J_i are the four faces of $\mathfrak{E}$, b_i is a one-cycle and $FF\mathfrak{E} = 1$. That these conditions on $F\mathfrak{E}$ can be satisfied is easily proved by an example (see no. 13 below). They can in fact be satisfied in many different ways for each $\mathfrak{E}$.

A μ_2 being thus defined for K, it remains to show that μ_2 is independent of the particular application of the rules just given for finding S from K. This is the result of theorems 6.1, 6.5 and 8.1. (Theorem 8.1 depends on 2.6.)

THEOREM 6.1. *Let x be a one-chain of S having property ϕ, outset $\bar{\omega}$ and finish ω^{-1}. Then if $\bar{S}$ is obtained from S by replacing $FJ = j$ in S by $\bar{F}J = xjx^{-1}$ in $\bar{S}$ for each 2-cell J, and, for each 3-cell $\mathfrak{E}$, replacing*

$$F\mathfrak{E} = (a_1 J_1 a_1^{-1}) \cdots (a_s J_s a_s^{-1}) \qquad \text{in } S \text{ by} \tag{6.2}$$

$$\bar{F}\mathfrak{E} = (b_1 J_1 b_1^{-1}) \cdots (b_s J_s b_s^{-1}) \qquad \text{in } \bar{S}, \text{ where} \tag{6.3}$$

$$b_i = x\, a_i x^{-1},$$

then $\mu_2(S) \approx \mu_2(\bar{S})$.

This theorem shows that μ_2 does not depend on the particular point ω.

PROOF OF THEOREM 6.1. Use a bar to distinguish the groups and boundary operator of $\bar{S}$. To each syllable $\Omega_i = a_i J_i a_i^{-1}$ of S assign the syllable

$$\tau\Omega_i = b_i J_i b_i^{-1} \text{ of } \bar{S}.$$

Then

$$\bar{F}(\tau\Omega_i) = xF\Omega_i x^{-1}. \tag{6.4}$$

If $\tau(\Omega_1\Omega_2 \cdots \Omega_s)$ is set equal to $(\tau\Omega_1)(\tau\Omega_2) \cdots (\tau\Omega_s)$, (6.4) shows that $\tau\mathfrak{Z}_2 = \bar{\mathfrak{Z}}_2$. Since $x(F\Omega)ax^{-1} = (xF\Omega x^{-1})(xax^{-1})$, (r.2) and (6.4) show that

$$\begin{aligned}\tau(F\Omega_1\,\Omega_2\,F\Omega_1^{-1}) &= (xF\Omega_1 x^{-1})\tau\Omega_2(xF\Omega_1^{-1}x^{-1}) \\ &= (\bar{F}\tau\Omega_1)\tau\Omega_2(\bar{F}\tau\Omega_1)^{-1} = \tau(\Omega_1\Omega_2\Omega_1^{-1}),\end{aligned}$$

so τC, $C \in \mathfrak{Z}_2$, is a unique element of $\bar{\mathfrak{Z}}_2$, and τ is a homorphism of $\mathfrak{Z}_2$ into $\bar{\mathfrak{Z}}_2$. Now by (6.2) and (6.3) $\tau F\mathfrak{E} = \bar{F}\mathfrak{E}$ and $\tau^{-1}\bar{F}\mathfrak{E} = F\mathfrak{E}$, whence $\mu_2 \approx \bar{\mu}_2$.

THEOREM 6.5. *If $\bar{S}$ is obtained from S by replacing, for a single 2-cell E of S, $FE = e$ by $\bar{F}E = xex^{-1}$ where $x \in \mathfrak{Z}_1$, and replacing E where it occurs in the boundary $F\mathfrak{E}$ of three-cells $\mathfrak{E}$ by $x^{-1}Ex$ in $\bar{F}\mathfrak{E}$, then $\mu_2(S) \approx \mu_2(\bar{S})$.*

Since replacing e by e^{-1} is a trivial change, this theorem shows that μ_2 is independent of the particular choices of the boundaries of the 2-cells. For instance if $FE = a(rst)a^{-1}$ and $\bar{F}E = b(str)b^{-1}$, putting $x = br^{-1}a^{-1}$ brings the change under 6.5.

PROOF OF THEOREM 6.5. In chains of $\bar{S}$ replace E wherever it occurs by $x\bar{E}x^{-1}$, where $\bar{E}$ is a new symbol such that $\bar{F}\bar{E} = e$. This trivial change of basis for the 2-chains of $\bar{S}$ makes the cycles and boundaries of $\bar{S}$ formally identical with those of S.

7. Theorem 8.1 below shows that μ_2 does not depend on the particular choice of the boundaries of the 3-cells made in determining S from K. The following purely algebraic lemmas are used in its proof.

LEMMA 7.1. *If in a free multiplicative group the element $E \neq 1$ contains, in normal form,* (N.C.pp. 497–8), *no repeated letter, then*

$$D^{-1}ED = E \tag{7.2}$$

implies that the element D is a power of E.

If $D = 1$ the lemma is trivial, so assume that $D \neq 1$, and suppose that D is given in normal form by

$$D = b_1 \cdots b_p c_{p+1} \cdots c_q b_p^{-1} \cdots b_1^{-1} \tag{7.3}$$

where $q > p$ and $c_{p+1}c_q \neq 1$. Suppose further that, in normal form,

$$E = a_1 a_2 \cdots a_n \tag{7.4}$$

where

$$a_i = a_j^{\pm 1} \quad \text{implies} \quad i = j. \tag{7.5}$$

Then

$$b_1 \cdots b_p c_q^{-1} \cdots c_{p+1}^{-1} b_p^{-1} \cdots b_1^{-1} a_1 \cdots a_n b_1 \cdots b_p c_{p+1} \cdots c_q b_p^{-1} \cdots b_1^{-1} = a_1 \cdots a_n . \tag{7.6}$$

Hence either, case 1, $a_1 = b_1$, or, case 2, $a_n^{-1} = b_1$, since the l.h.s. of (7.6) must be reducible to the r.h.s. which is normal.

Case 1. If $a_1 = b_1$, (7.6) yields

$$b_2 \cdots b_p c_q^{-1} \cdots c_{p+1}^{-1} b_p^{-1} \cdots b_2^{-1} a_2 \cdots a_n a_1 b_2 \cdots b_p c_{p+1} \cdots c_q b_p^{-1} \cdots b_2^{-1} = a_2 \cdots a_n a_1 \tag{7.7}$$

and for the next step in the reduction of the l.h.s. it is necessary that either $a_2 = b_2$ or $a_1^{-1} = b_2$. But if $a_1^{-1} = b_2$, then $b_1 = b_2^{-1}$ which contradicts the normality of (7.3). Hence $a_2 = b_2$ and another step similar to that from (7.6) to (7.7) is possible and so on. After p steps there result

$$b_i = a_j, \qquad 1 \leqq i \leqq p, \qquad i \equiv j \bmod n, \tag{7.8}$$

and

$$c_q^{-1} \cdots c_{p+1}^{-1} a_{r+1} \cdots a_n a_1 \cdots a_r c_{p+1} \cdots c_q = a_{r+1} \cdots a_n a_1 \cdots a_r \tag{7.9}$$

where

$$p = sn + r, \qquad s \text{ integral}, \qquad 0 \leqq r < n. \tag{7.10}$$

Again normality of (7.3) gives $a_r \neq c_{p+1}^{-1}$, so

$$c_{p+1} = a_{r+1} \tag{7.11}$$

and

$$c_{p+1}^{-1}c_q^{-1}\cdots c_{p+2}^{-1}a_{r+2}\cdots a_n a_1\cdots a_r a_{r+1}c_{p+2}\cdots c_p c_{p+1} = a_{r+2}\cdots a_n a_1\cdots a_{r+1}. \tag{7.11.1}$$

Then $c_{p+2} = a_{r+2}$ and a further step is possible. After $q - p$ such steps

$$c_i = a_j, \qquad p < i \leqq q, \qquad i \equiv j \bmod n, \tag{7.12}$$

and

$$c_q^{-1}\cdots c_{p+1}^{-1}a_{m+1}\cdots a_n a_1\cdots a_m c_{p+1}\cdots c_q = a_{m+1}\cdots a_n a_1\cdots a_m \tag{7.13}$$

where

$$q = kn + m, \qquad k \text{ integral}, \qquad 0 \leqq m < n. \tag{7.14}$$

So by (7.14) $c_{p+1} = a_{m+1}$. But by (7.11) $c_{p+1} = a_{r+1}$ so, using (7.5) $m = r$ and by (7.10) and (7.14)

$$q - p \text{ is a multiple of } n. \tag{7.15}$$

Using (7.8) and (7.12) with (7.15)

$$b_p = a_r = a_m = c_q. \tag{7.16}$$

But if $p \neq 0$, (7.16) contradicts the normality of (7.3), so $p = 0$ and by (7.15) q is a multiple of n. This with (7.12) proves the lemma in case 1.

CASE 2. Replacing (7.2) by $D^{-1}E^{-1}D = E^{-1}$ and renaming the letters of the word E so that $E = a_n^{-1}a_{n-1}^{-1}\cdots a_1^{-1}$ makes case 2 formally the same as case 1 and so provides, in case 2, a proof of the lemma for E^{-1} and hence for E.

LEMMA 7.17. *Lemma* 7.1 *holds if* (7.2) *is replaced by* $D^{-1}E\,D = E^{-1}$.

The proof is just as before except that in the formulas whose numbers follow the right hand side must be replaced by its inverse: (7.6), (7.7), (7.11.1), (7.13).

COROLLARY 7.18. *If* E *is as in lemma* 7.0, $D^{-1}E\,D = E^{-1}$ *implies* $E = 1$.

8. THEOREM 8.1. *If* $\bar{S}$ *is obtained from* S *by replacing*

$$F\mathfrak{E} = (b_1J_1b_1^{-1})\cdots(b_4J_4b_4^{-1}) \text{ in } S,$$

where $\mathfrak{E}$ *is a* 3*-cell of* S, J_i *is a* 2*-cell of* S *or the inverse of such a* 2*-cell by*

$$\bar{F}\mathfrak{E} = (c_1J_{k_1}^{r_1}c_1^{-1})\cdots(c_4J_{k_4}^{r_4}c_4^{-1}), \tag{8.2}$$

where k_1, k_2, k_3, k_4 *is a permutation of* 1, 2, 3, 4 *and* $r_i = \pm 1$, *then* $\mu_2 \simeq \bar{\mu}_2$.

PROOF. Here the groups $\mathfrak{Z}_2$ and $\bar{\mathfrak{Z}}_2$ are identical so the proof can be made by showing that $\mathfrak{F}_2 \equiv \bar{\mathfrak{F}}_2$. For this it is sufficient to show that $F\mathfrak{E} \in \bar{\mathfrak{F}}_2$ and $\bar{F}\mathfrak{E} \in \mathfrak{F}_2$, which can be done by finding one-cycles p and q such that

$$(p\,F\mathfrak{E}p^{-1})(q\,\bar{F}\mathfrak{E}^{\pm 1}q^{-1}) = 1, \tag{8.3}$$

for then

$$F\mathfrak{E} = p^{-1}q\bar{F}\mathfrak{E}^{\mp 1}q^{-1}p \in \bar{\mathfrak{F}}_2$$

and

$$\bar{F}\mathfrak{E} = q^{-1}pF\mathfrak{E}^{\mp 1}p^{-1}q \in \mathfrak{F}_2.$$

Suppose that $r_1 = 1$. (If it were -1 the notation could be changed by replacing $\bar{F}\mathfrak{E}$ by its inverse). Let

$$U = (b_2 J_2 b_2^{-1})(b_3 J_3 b_3^{-1})(b_4 J_4 b_4^{-1})$$

and write $FU = u$. Take a such that $k_a = 1$ in the permutation k_1, k_2, k_3, k_4.

Let $V = (c_1 J_{k_1}^{r_1} c_1^{-1}) \cdots (c_{a-1} J_{k_{a-1}}^{r_{a-1}} c_{a-1}^{-1})$ if $a > 1$

$V = 1$ if $a = 1$.

Let $W = (c_{a+1} J_{k_{a+1}}^{r_{a+1}} c_{a+1}^{-1}) \cdots (c_4 J_{k_4}^{r_4} c_4^{-1})$ if $a < 4$

$W = 1$ if $a = 4$.

Write $FV = v$ and $FW = w$. Then

$$F\mathfrak{E} = b_1 J_1 b_1^{-1} U$$

$$\bar{F}\mathfrak{E} = V c_1 J_1 c_1^{-1} W = V c_1 J_1 c_1^{-1} V^{-1} V W = v c_1 J_1 c_1^{-1} v^{-1} V W.$$

Let

$$G = (b_1^{-1} F\mathfrak{E}^{-1} b_1)(c_1^{-1} v^{-1} \bar{F}\mathfrak{E} v c_1). \tag{8.4}$$

Comparing (8.4) and (8.3) shows that a proof that $G = 1$ will prove theorem 8.1. But $G = (b_1^{-1} U^{-1} b_1)(c_1^{-1} v^{-1} V W v c_1)$ contains no 2-cells except J_2, J_3 and J_4. Hence G is a chain of the subsystem $\hat{S}$ of S containing the zero and one-cells of S and J_2, J_3, J_4 but no other 2-cells and no 3-cells at all. By the inverse of subdivision, using theorems 6.1 and 6.5 as often as necessary, $\hat{S}$ can be replaced by S' in which J_2, J_3, J_4 are united to form a single 2-cell E. Theorems 2.5, 6.1, 6.5 show that $\mu_2(\hat{S}) \approx \mu_2(S')$. So proving that $\mu_2(S') \approx 1$ will show that $\mu_2(\hat{S}) \approx 1$ and hence that the G of (8.4) considered as a chain of $\hat{S}$ is in $\hat{\mathcal{F}}_2$. But $\hat{S}$ has no 3-cells so this is tantamount to proving that $G = 1$ in $\hat{S}$. Finally, since $\hat{S}$ is a subsystem of S, $G = 1$ in $\hat{S}$ implies $G = 1$ in S. So everything hinges on showing $\mu_2(S') \approx 1$, which is the result of the following theorem.

9. THEOREM 9.1. *If S is a system having a single 2-cell E and no 3-cells, then $\mu_2(S) \approx 1$.*

PROOF. By theorem 6.1 there is no loss of generality in assuming that ω is the outset of FE. Let $FE = e = xyz$, where x, y, z are one-cells of S or their inverses. Since E is a simplex, x, y and z are distinct. Theorem 9.1 is equivalent to the following lemma.

LEMMA 9.2. *If*

$$D \equiv (a_1 E^{q_1} a_1^{-1}) \cdots (a_s E^{q_s} a_s^{-1}) \qquad \text{where } q_i = \pm 1, \tag{9.3}$$

defines an element of $\mathfrak{Z}_2$ for the S now under consideration, then $D = 1$.

PROOF. Suppose the a_i of (9.3) are in normal form. If

$$a_i = b e^r c, \qquad r = \pm 1, \tag{9.4}$$

then

$$(a_iE^{q_i}a_i^{-1}) = (bE^rb^{-1})(bcE^{q_i}c^{-1}b^{-1})(bE^{-r}b^{-1}),$$

so it may be assumed that no a_i in (9.3) can be written in the form (9.4). Now since x and its inverse must occur equally often in (9.3) (because $FD = 1$) s must be even. If $s = 2$, $FD = (a_1e^{q_1}a_1^{-1})(a_2e^{q_2}a_2^{-1}) = 1$ so $a_2^{-1}a_1e^{q_1}a_1^{-1}a_2 = e^{-q_1}$ and, by either lemma 7.1 or 7.17, $a_2^{-1}a_1 = e^t$ and $q_1 = -q_2$. Therefore $D = (a_2E^ta_2^{-1})(a_2E^{q_1}a_2^{-1})(a_2E^{-t}a_2^{-1})(a_2E^{-q_1}a_2^{-1}) = 1$ which proves lemma 9.2 for $s = 2$.

As hypothesis of an induction assume that S is such that all 2-cycles of S (having no a_i in form (9.4)) which can be written in less than s syllables define the identity of $\mathfrak{Z}_2$, i.e. if H is a cycle of less than s syllables, $H = 1$. This hypothesis has just been proved for $s = 3$. The D of (9.3) represents a cycle of s syllables. Suppose one of these syllables, Ω, contains a one-cell m which is not $x^{\pm1}$, $y^{\pm1}$ or $z^{\pm1}$:

(9.4.1) $$\Omega = bmcE^qc^{-1}m^{-1}b^{-1} \qquad \text{where } c \text{ contains no } m.$$

Since $ce^qc^{-1} = 1$ implies $e = 1$ which is impossible, the m in Ω must, in $FD = 1$, cancel the m in some other syllable of D, i.e. for some Ω

(9.5) $$D \equiv M(bmcE^qc^{-1}m^{-1}b^{-1})\ N\ (dmgE^rg^{-1}m^{-1}d^{-1})\ T$$

where M, N, T are 2-chains, $r = \pm1$ and

(9.6) $$b^{-1}nd = 1 \qquad \text{when } FN = n,$$

for otherwise $FD \neq 1$. Using (9.6) in (9.5)

(9.7) $$D = M[bm(cE^qc^{-1})(gE^rg^{-1})m^{-1}b^{-1}]NT.$$

If the boundary of the chain in the square brackets is 1, $D = 1$ by hypothesis of the induction. If not, the bracket can itself play the role of the expression (9.4.1) and the steps (9.5)–(9.7) can be repeated for the same m until either the hypothesis of the induction can be used or $D = pm^tD'm^{-t}p^{-1}$, where $D' = (h_1E^{r_1}h_1^{-1}) \cdots (h_sE^{r_s}h_s^{-1})$, $r_i = \pm1$, $FD' = 1$. Though D' is not necessarily a chain, it is convenient to introduce it as a symbol (see definition 1.7) and to define FD' in the obvious way. Now D' contains m fewer times than D, all h_i have property ϕ, a common outset and the finish ω^{-1}.

If D' has a letter m' which is not $x^{\pm1}$, $y^{\pm1}$ or $z^{\pm1}$, the same process may be applied to m' in D' as was to m in D. Since $pmD'm^{-1}p^{-1}$ is a cycle of S the same use of the hypothesis of induction can be made at the new steps as was made before. Eventually either lemma 9.1 is verified or $D = u\bar{D}u^{-1}$, where $\bar{D} = (k_1E^{p_1}k_1^{-1}) \cdots (k_sE^{p_s}k_s^{-1})$ and $p_i = \pm1$ and all k_i have property ϕ, a common outset σ and finish ω^{-1} and are made up of the letters x, y z. Like D', $\bar{D}$ is not necessarily a chain.

Suppose σ is the outset of y. Then xk_i is a cycle on ω and so is a one-cycle of S and must, in normal form, be $(xyz)^{a_i} = e^{a_i}$, a_i an integer. Therefore $x\bar{D}x^{-1}$ is a chain such that

$$x\bar{D}x^{-1} = (e^{a_1}E^{p_1}e^{-a_1}) \cdots (e^{a_s}E^{p_s}e^{-a_s}) = E^{a_1}E^{p_1}E^{-a_1} \cdots E^{a_s}E^{p_s}E^{-a_s}$$

is a cycle of S and so $F(x\bar{D}x^{-1}) = 1$ which implies $p_1 + p_2 + p_3 + \cdots + p_s = 0$ which in turn implies $x\bar{D}x^{-1} = 1$. But $D = ux^{-1}(x\bar{D}x^{-1})xu^{-1} = ux^{-1}(1)xu^{-1} = 1$ by (r.3). If σ were the outset of z, similar reasoning would give $xy\bar{D}y^{-1}x^{-1} = 1$ and $D = 1$ again. If σ were ω, then $\bar{D} = 1$ directly and so $D = 1$. Now the hypothesis of the induction has been proved for s on the assumption that it holds for $s - 1$, so the proof of lemma 9.1 and hence of theorem 8.1 is complete.

10. In this and subsequent sections the theory just developed will be applied to some particular simplicial complexes. By corollary 2.6, the n-sphere S^n can be considered to be composed of two n-simplexes, $Y^n = \{\alpha_0\alpha_1 \cdots \alpha_n\}$ and $Z^n = \{\beta_0\beta_1 \cdots \beta_n\}$ where the identifications of lower dimensional simplexes given by $\{\alpha_{i_0}\alpha_{i_1} \cdots \alpha_{i_p}\} = \{\beta_{i_0}\beta_{i_1} \cdots \beta_{i_p}\}^{-1}$, $p < n$, have been made. So, n for > 2, the two-cycles of S^n are the two-cycles of the n-simplex Y^n and therefore $\mu_2(Y^n) \approx 1$ implies $\mu_2(S^n) \approx 1$. But for $n > 2$, $\mu_2(S^n) \approx 1$ implies $\mu_2(Y^{n+1}) \approx 1$. It follows that $\mu_2(Y^3) \approx 1$ implies $\mu_2(S^n) \approx 1$ for $n > 2$.

Now S^2 and $Y^3 = \mathcal{J}$ can be treated together. Adopt the notation:

$$e_0 = \{\alpha_1\alpha_2\} = \{\beta_1\beta_2\}^{-1}, \qquad e_1 = \{\alpha_0\alpha_2\} = \{\beta_0\beta_2\}^{-1}, \qquad e_2 = \{\alpha_0\alpha_1\} = \{\beta_0\beta_1\}^{-1},$$

$$\alpha_i = \{\alpha_i\} = \{\beta_i\}^{-1}, \qquad \omega = \alpha_0,$$

$$Fe_0 = \alpha_1\alpha_2^{-1}, \qquad Fe_1 = \alpha_0\alpha_2^{-1}, \qquad Fe_2 = \alpha_0\alpha_1^{-1},$$

$$FY = e_2e_0e_1^{-1}, \qquad FZ = e_1e_0^{-1}e_2^{-1}, \qquad F\mathcal{J} = YZ.$$

By theorem 2.5, Y and Z and e_1 can be replaced by E where $FE = 1$ and $F\mathcal{J} = E$. Then e_2 and e_0 can be replaced by e where $Fe = \alpha_0\alpha_2^{-1}$. So it follows that the cycles of S^2 and $\mathcal{J}$ are E^t, and so $\mu_2(S^2)$ is the infinite cyclic group whereas $\mu_2(\mathcal{J}) \approx 1$. It follows that $\mu_2(S^n) \approx 1$ for $n > 2$. Since 1 is the only 2-cell of S^1 and S^0 it follows that $\mu_2(S^n) \approx 1$ for $n < 2$.

11. Let M be the complex composed of S^2 and an S^1 (called c) having the single vertex α_0 in common with S^2 (see no. 10). $Fc = \alpha_0\alpha_0^{-1} = 1$. Then $\mu_2(M)$ is the free abelian group with the infinite set c^tEc^{-t}, t integral, of generators. (Notice that $(c^tEc^{-t})(c^sE^{-1}c^{-s}) = 1$ implies $t = s$.)

Let N be a complex like M except that c bounds a 2-cell X: $FX = c$. Then $c^tEc^{-t} = X^tEX^{-t} = X^tX^{-t}E$ by theorem 1.8, whence $c^tEc^{-t} = E$. Therefore $\mu_2(N)$ is the infinite cyclic group.

12. Let K be the torus with vertex ω, one-cells a and b, and 2-cell E, where $Fa = Fb = \omega\omega^{-1}$, $FE = a^{-1}b^{-1}ab$. The only 2-cycle is the identity so μ_2 is the identity for the torus.

13. Lens spaces.[6] Take p 3-simplexes $\mathfrak{G}_0, \mathfrak{G}_1, \cdots, \mathfrak{G}_{p-1}$ such that $\mathfrak{G}_i$

[6] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Berlin 1934, p. 210.

has on its boundary the 2-simplexes A_i, B_i, C_i, D_i and the one-simplexes a_i, b_i, c_i, d_i, e_i, f_i with boundary relations:

$$F\mathfrak{G}_i = A_iB_iD_iC_i, \qquad FA_i = a_ic_i^{-1}d_i, \qquad FB_i = d_i^{-1}b_ie_i,$$
$$FC_i = e_i^{-1}f_ia_i^{-1}, \qquad FD_i = e_i^{-1}(b_i^{-1}c_if_i^{-1})e_i.$$

Group the tetrahedra radially about the one-simplex $b = b_0 = b_1 = \cdots = b_{p-1}$ so that

$$f_i = e_{i+1}, \qquad c_i = d_{i+1}, \qquad D_i = e_i^{-1}f_iB_{i+1}^{-1}f_i^{-1}e_i \tag{13.1}$$

if subscripts are reduced mod p. Thus grouped the tetrahedra form a "lens". The simplexes C_i compose one face of this lens, the simplexes A_i the other. If p and q are relatively prime and $q < p/2$, subject one lens face to a rotation of $2\pi q/p$ and identify it with the other. This amounts to letting

$$a_i = a_{i+q}, \qquad e_i = d_{i+q}, \qquad f_i = c_{i+q}, \qquad C_i = A_{i+q}^{-1}. \tag{13.2}$$

Since the numbers kq, $k = 0, 1, \cdots, p-1$, are distinct mod p, the effect of (13.1) and (13.2) on the one-simplexes can be summed up in

$$a_i = a, \qquad b_i = b, \qquad e_i = f_{i-1} = c_{i+q-1} = d_{i+q} = i.$$

The number i on the extreme right is to be understood as an abbreviation for r_i where r is a fixed letter and i is a variable subscript. Subscripts are to be reduced mod p.

Then for the lens space (p, q),

$$\begin{cases} F\mathfrak{G}_i = A_iB_i[i^{-1}(i+1)B_{i+1}^{-1}(i+1)^{-1}i]A_{i+q}^{-1} \\ FA_i = a(i-q+1)^{-1}(i-q) \\ FB_i = (i-q)^{-1}bi. \end{cases} \tag{13.3}$$

The system S determined by the equations (13.3) (plus easily deduced boundary relations for the one-cells, here omitted) is now simplified by repeated use of theorem 2.5. First let

$$X_i = i^{-1}(i+1)B_{i+1}^{-1}(i+1)^{-1}i. \tag{13.4}$$

I. Replace $\mathfrak{G}_0$, $\mathfrak{G}_q$ and A_q by $\mathfrak{E}_1$ where

$$F\mathfrak{E}_1 = F(\mathfrak{G}_0\mathfrak{G}_q) = A_0B_0X_0B_qX_qA_{2q}^{-1}.$$

Replace $\mathfrak{E}_1$, $\mathfrak{G}_{2q}$ and A_{2q} by $\mathfrak{E}_2$ where

$$F\mathfrak{E}_2 = F(\mathfrak{E}_1\mathfrak{G}_{2q}) = A_0B_0X_0B_qX_qB_{2q}X_{2q}A_{3q}^{-1},$$

and so on up to and including the replacement of $\mathfrak{E}_{p-3}$, $\mathfrak{G}_{(p-2)q}$ and $A_{(p-2)q}$ by $\mathfrak{E}_{p-2}$ where

$$F\mathfrak{E}_{p-2} = F(\mathfrak{E}_{p-3}\mathfrak{G}_{(p-2)q}) = A_0(B_0X_0 \cdots B_{(p-2)q}X_{(p-2)q})A_{(p-1)q}^{-1}.$$

Replace A_0, $A_{(p-1)q}$ and a by J where $FJ = F(A_{(p-1)q}^{-1}A_0)$ and

$$F\mathfrak{E}_{p-2} = A_0[(B_0X_0 \cdots B_{(p-2)q}X_{(p-2)q})A^{-1}_{(p-1)q}A_0]A_0^{-1}$$
$$= (B_0X_0 \cdots B_{(p-2)q}X_{(p-2)q})J$$

by theorem 1.8 since the square bracket is a cycle; and, similarly,

$$F\mathfrak{G}_{(p-1)q} = J^{-1}B_{(p-1)q}X_{(p-1)q}.$$

Replace $\mathfrak{E}_{p-2}$, $\mathfrak{G}_{(p-1)q}$ and J by $\mathfrak{E}$ where

$$F\mathfrak{E} = B_0X_0B_qX_q \cdots B_{(p-1)q}X_{(p-1)q}. \tag{13.5}$$

There now remain of the 3-cells only $\mathfrak{E}$, of the 2-cells only the B_i, and of the one-cells b and i, $i = 0, 1, \cdots, (p-1)$.

II. Now it is possible to eliminate all but one 2-cell and all the one-cells but b and $[(p-1)q]$, which last will also be written $(-q)$. Let

$$Z_{kq} = (-q)^{-1}b^{1+k-p}(1+kq)B^{-1}_{1+kq}(1+kq)^{-1}b^{p-k-1}(-q). \tag{13.6}$$

Since it is an immediate consequence of (13.3) that

$$F(B_{(k+1)q} \cdots B_{(p-1)q}) = (kq)^{-1}b^{p-k-1}(-q) \tag{13.7}$$

using (13.7) on (13.6) and then (r.2) on the r.h.s. of the result gives, by (13.4),

$$X_{kq} = B_{(k+1)q} \cdots B_{(p-1)q}Z_{kq}B^{-1}_{(p-1)q} \cdots B^{-1}_{(k+1)q}. \tag{13.8}$$

Now using (13.8) to eliminate the X's from (13.5) gives

$$F\mathfrak{E} = B_0B_q \cdots B_{(p-1)q}Z_0Z_q \cdots Z_{(p-1)q}. \tag{13.9}$$

Let D_{rq} be defined by

$$D_{rq} = B_1B_{1+q} \cdots B_{1+rq}. \tag{13.10}$$

LEMMA. $Z_0Z_q \cdots Z_{rq} = (-q)^{-1}b^{1+r-p}(1+rq)D^{-1}_{rq}(1+rq)^{-1}b^{p-r-1}(-q)$.

Proof by induction on r. For $r = 0$, this is (13.6), so assume the lemma for $r < k$. Let

$$S = (Z_0 \cdots Z_{(k-1)q})Z_{kq} = (-q)^{-1}\{[b^{k-p}(1+(k-1)q)D^{-1}_{(k-1)q}(1+(k-1)q)^{-1}b^{p-k}]$$
$$\cdot[b^{1+k-p}(1+kq)B^{-1}_{1+kq}(1+kq)^{-1}b^{p-k-1}]\}(-q).$$

By (13.3), $(1+(k-1)q) = b(1+kq)(FB_{1+kq})^{-1}$ so

$$S = (-q)^{-1}b^{k-p+1}(1+kq)D^{-1}_{kq}(1+kq)^{-1}b^{p-k-1}(-q),$$

which proves the lemma.

It is a consequence of the lemma that

$$Z_0Z_q \cdots Z_{(p-1)q} = (-q)^{-1}(1+(p-1)q)D^{-1}_{(p-1)q}(1+(p-1)q)^{-1}(-q). \tag{13.11}$$

$$\text{If} \qquad 1+nq \equiv 0 \bmod p \tag{13.12}$$

then

$$1+(p-1)q \equiv (p-n-1)q \bmod p$$

and

$$1 + (n - k)q \equiv (p - k)q \bmod p$$

for every k, so, by (13.10)

$$D_{(p-1)q} = B_1 \cdots B_{1+(n-1)q}B_0B_q \cdots B_{(p-n-1)q}B_{(p-n)q} \cdots B_{(p-1)q}B_{1+(n-1)q}^{-1} \cdots B_1^{-1}. \tag{13.13}$$

But $F(B_1 \cdots B_{1+(n-1)q}) = (1 - q)^{-1}b^n(1 + (n - 1)q)$, so (13.13) becomes

$$D_{(p-1)q} = (1 - q)^{-1}b^n(1 + (n - 1)q)B_0B_q \cdots B_{(p-1)q}(1 + (n - 1)q)^{-1}b^{-n}(1 - q). \tag{13.14}$$

Now combining (13.14), (13.11) and (13.9) and noticing that $(1 - q) \equiv (1 + (p - 1)q) \bmod p$ yields

$$F\mathfrak{E} = B_0B_q \cdots B_{(p-1)q}[(-q)^{-1}b^n(-q)(B_0B_q \cdots B_{(p-1)q})^{-1}(-q)^{-1}b^{-n}(-q)]. \tag{13.15}$$

The elimination of cells now takes place in $(p - 1)$ steps. Let $G_0 = B_0$ and at the k^{th} step replace G_{k-1} and B_{kq} and $(kq - q)$ by G_k, where $FG_k = F(G_{k-1}B_{kq})$. Then letting $G_{(p-1)q} = G$ and $(-q)^{-1}b(-q) = y$, (13.15) becomes

$$F\mathfrak{E} = G(y^nG^{-1}y^{-n}), \tag{13.16}$$

and, from the definition of G

$$FG = y^p. \tag{13.17}$$

From (13.17) it follows that π_1 is the cyclic group of order p. Notice that by (r.2) and (13.17)

$$(y^{r+p}G^ty^{-r-p}) = y^rG^ty^{-r}.$$

Any 2-cycle, H, is a combination of syllables of the type (y^rGy^{-r}), and if $H \neq 1$, somewhere in H there must be an adjacent pair

$$M = (y^sGy^{-s})(y^tG^{-1}y^{-t})$$

or the inverse of such a pair, $s \not\equiv t \bmod p$, so that $H = LMN$. But by (13.16),

$$M(y^tF\mathfrak{E}y^{-t}) = (y^sGy^{-s})(y^{n+t}G^{-1}y^{-n-t}).$$

If $n + t \equiv s \bmod p$ this shows that $M \in \mathfrak{F}_2$. If $n + t \not\equiv s \bmod p$,

$$M(y^tF\mathfrak{E}y^{-t})(y^{n+t}F\mathfrak{E}y^{-n-t}) = (y^sGy^{-s})(y^{2n+t}G^{-1}y^{-2n-t}).$$

If $2n + t \equiv s \bmod p$ this shows that $M \in \mathfrak{F}_2$. Otherwise a third step is made and so on. But by (13.12), n and p are relatively prime. Hence there is a k such that $kn + t \equiv s \bmod p$, and so after k steps it results that $M \in \mathfrak{F}_2$. Hence H and LN determine the same element of μ_2. Now the argument applied to H can be applied to LN and so on until eventually it follows that H defines the same element of μ_2 as the identity. This shows that for the lens space (p, q), μ_2 is the identity.

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SOME REMARKS ON SET THEORY

BY P. ERDÖS

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This paper contains a few disconnected results on the theory of sets.

I. Sierpinski[1] proved that under the assumption of the continuum hypothesis there exists a single valued function $f(x)$ whose inverse function is also single valued and which maps the sets of measure 0 into sets of first category and whose inverse function maps the sets of first category into sets of measure 0. He stated the problem[2] whether a function exists which has the above property and also the following one: It maps the sets of first category into sets of measure 0 and its inverse function maps the sets of measure 0 into sets of first category. Thus the function would interchange the sets of measure 0 and the sets of first category. We shall prove that such a function exists. Our proof will be very similar to that of Sierpinski: we will of course assume that the continuum hypothesis holds.

Construction of $f(x)$: It can be shown[3] that a transfinite sequence G_α of G_δ sets of measure 0 and a transfinite sequence F_α of F_σ sets of first category exists ($\alpha < \Omega_1$, Ω_1 is the first ordinal number of the third number class) having the following properties: 1) $G_1 \cup F_1 = R$, $G_1 \cap F_1 = \Lambda$ (R denotes the set of all real numbers), 2) every set of measure 0 is contained in some G_α and every set of first category is contained in some F_α 3) $A_\alpha = G_\alpha - G_\alpha \cap \bigcup_{\beta<\alpha} G_\beta$, $B_\alpha = F_\alpha - F_\alpha \cap \bigcup_{\beta<\alpha} F_\beta$ both have the power of the continuum, for every α. We evidently have $G_1 = \bigcup_{\alpha>1} B_\alpha$, $F_1 = \bigcup_{\alpha>1} A_\alpha$. Hence we can construct a function $f(x)$ in such a way that $f(A_\alpha) = B_\alpha$ for every $\alpha > 1$, and that $f\,f(x) = x$ for every x. The function $f(x)$ is clearly a single valued function whose inverse $f^{-1}(x)$ is also single valued. Since, in addition $f(x)$ coincides with its inverse, we have only to show that $f(x)$ maps both the sets of measure 0 onto sets of first category, and the sets of first category onto sets of measure 0. But both of these statements are obvious. For let G be any set of measure 0; by assumption $G \subset G_\alpha$ for some α and $f(G) \subset \bigcup_{\beta<\alpha} F_\beta$, which is a set of first category. Similarly let F be any set of first category; by assumption $F \subset F_\alpha$ for some α, and $f(F) \subset \bigcup_{\beta<\alpha} G_\beta$ which is a set of measure 0: This completes the proof.

II. Let m be a cardinal number. Two sets A and B in Euclidean space are called m-equivalent if they can be split into m summands $A = UA_\alpha$, $B = UB_\alpha$, $A_{\alpha_i} \cap A_{\alpha_j} = B_{\alpha_i} \cap B_{\alpha_j} = 0$, and $A_\alpha \cong B_\alpha$. (The sign $\cong$ denotes congruence.)

[1] W. Sierpinski, Fund. Math. Vol. 22, p. 276–278.
[2] Ibid.
[3] Ibid.

Banach and Tarski[4] proved that in three space any two sets containing open sets are finitely equivalent, and that on the line and the plane any two sets containing open sets are countably equivalent.

Professor Tarski[5] communicated to me the following result of Lindenbaum: There exist 2^c linear sets no two of which are countably equivalent. This result was never published, and Tarski does not remember the details of the proof. I have succeeded in proving that if m is any cardinal number $< c$, then there exist 2^c linear sets no two of which are m-equivalent. I do not know whether my proof differs from that of Lindenbaum, but I have thought it might be worth publishing, since the result has some interesting applications.

First we remark that it is easy to construct 2^n subsets of an infinite set A of power n such that the symmetric difference $(x - y) \cup (y - x)$ of any two subsets x and y has the power n. It is sufficient to divide A into n mutually exclusive subsets of power n, and to consider the unions of all these subsets.

Let now $\{a_\alpha\}$ be a Hamel base ($\alpha < \omega_\xi$, ω_ξ is the smallest ordinal belonging to the power of the continuum.) and let $A_\beta (\beta < \omega_\eta$, ω_η the smallest ordinal belonging to 2^c) be a family of subsets of this Hamel base such that the symmetric difference $(A_{\beta_1} - A_{\beta_2}) \cup (A_{\beta_2} - A_{\beta_1})$ has always the power c. Denote by U_β the set of real numbers of the form $\sum c_k a_k$ where the c_k are rational numbers and the a_k belong to A_β. Now we show that for $\beta_1 \neq \beta_2$ U_{β_1} and U_{β_2} are not m-equivalent. We can clearly assume that A_{β_2} contains c elements not contained in A_{β_1}. A being a set of numbers and x an arbitrary number, let us denote by $A + x$ the set of all numbers $z + x$ where z belongs to A. Also we denote by $A^{(y)}$ the reflection of A with respect to y. It suffices to show that if $\{x_\mathfrak{S}\}$ and $\{y_\mathfrak{S}\}$ are two sets of power m ($\mathfrak{S} < \omega_\delta$, ω_δ is the smallest ordinal number belonging to m) then the union of all the sets $U_{\beta_1} + x_\mathfrak{S}$, $U_{\beta_1}^{(y_\mathfrak{S})}$ does not contain U_{β_2}. And this is clear for if we denote by a_ξ the elements of the Hamel base necessary to express the $x_\mathfrak{S}$ and the $y_\mathfrak{S}$ (the power of the a_ξ is clearly $\leqq m$) our set $\bigcup U_{\beta_1} + x_\mathfrak{S}$, $U_{\beta_1}^{(y_\mathfrak{S})}$ can therefore be generated by the elements of A_{β_1} and by at most m other elements of the Hamel basis; while U_{β_2} is generated by the elements of A_{β_2}, and the latter set contains c elements which do not belong to A_{β_1}. This completes our proof.

III. A set B of real numbers is said to be of absolute measure 0 if it is finitely equivalent to a subset of an arbitrarily small interval. It is said to be of absolute measure α if for every ϵ it is finitely equivalent to a subset of an interval of length $\alpha + \epsilon$, and a subset of it is finitely equivalent with the interval of length $\alpha - \epsilon$.[6]

It is well known that the power of Lebesgue measurable sets mod null sets is of power c, but that the power of Lebesgue measurable sets is 2^c. Tarski[7]

[4] Banach and Tarski, Fund. Math. Vol. 6, p. 244–278.

[5] Oral communication.

[6] Tarski, Fund. Math. Vol. 30, p. 218–253. This paper contains the definition and all the properties used of absolute measure used in this proof.

[7] Oral communication.

posed the problem: What is the power of absolutely measurable sets mod sets of absolute measure 0? (It is of course clear the power of absolutely measurable sets is 2^c.)[8] We are going to prove that the power in question is 2^c.

First it is clear that it suffices to prove that the power of all sets in the interval (0, 1) mod sets of absolute measure 0 is of power 2^c. For if we take any set A in (0, 1) and translate it by 1, take its complement in (1, 2) denote it by B, then $A + B$ has absolute measure 1, hence if A_1 and A_2 are not congruent mod sets of absolute measure 0, $A_1 + B_1$ and $A_2 + B_2$ are also not congruent. This is a strong indication of the truth of our theorem, since it is well known that the power of all sets mod sets of Lebesgue measure 0 is also 2^c.

To prove our theorem it clearly suffices to show that there exist in the interval (0, 1) c disjoint sets whose absolute measure is not 0, for by taking all possible sums of these sets we clearly get 2^c sets no two of which are congruent mod sets of absolute measure 0.

Let now $\{a_\alpha\}$ be a Hamel base with $a_1 = 1$. Split it into c disjoint sets of power c. Denote these sets by V_β. We define the sets R_β as follows: $x \in R_\beta$ if and only if $0 \leqq x \leqq 1$ and $x = \sum_{i=1}^{k} c_i a_{\alpha_i}$, the c_i rational and different from 0 and $\alpha_1 < \alpha_2 < \cdots \alpha_k$, $a_{\alpha_k} V_\beta$. (For $i < k$ a_{α_i} does not have to belong to V_β.) We are going to prove that the disjoint sets R_β are not of absolute measure 0. In fact we shall show that R_β is not finitely equivalent with any subset of $(0, \frac{1}{2})$. For suppose that R_β is finitely equivalent with a subset of $(0, \frac{1}{2})$. This would mean that there exist sets $U_1, U_2, \cdots, U_r$ whose sum is the interval $(0, \frac{1}{2})$, and real numbers $x_1, x_2, \cdots x_j, y_1, y_2, \cdots y_l$, $k + l = r$, such that, R_β is contained in $\bigcup U_i + x_i$, $U_j^{(y_j)}$, the sets U_i are supposed to be mutually exclusive. Let a_k be the a_α of largest index which occurs in the representation of the x_i and y_j and denote by R'_β those elements of R'_β in whose representation the a_α of largest index has an index $> k$. Then if w is an element of R'_β and $w \in U_i + x_i$ there exists a $z \in U_i$ with $z = w - x_i$, hence $z \in R'_\beta$, also if $w \in U_j^{(y_j)}$ there again exists a $z \in U_j$ with $z = 2y_j - w$ hence $z \in R'_\beta$. Similarly if $z \in R'_\beta$ we have $z + x_i \in R'_\beta$ and $2y_j - z \in R'_\beta$. Thus we see that $(R'_\beta \cap U_i) + x_i = R'_\beta \cap (U_i + x_i)$ and $(R'_\beta \cap U_j)^{y_j} = R'_\beta \cap U_j^{y_j}$, hence we conclude that $R'_\beta \cap (0, \frac{1}{2})$ is finitely equivalent to $R'_\beta = R'_\beta \cap (0, \frac{1}{2}) \cup R'_\beta \cap (\frac{1}{2}, 1)$. On the other hand a translation by $\frac{a_1}{2} = \frac{1}{2}$ shows that $R'_\beta \cap (0, \frac{1}{2}) \cong (R'_\beta \cap (0, \frac{1}{2})) + \frac{1}{2} = R'_\beta \cap (\frac{1}{2}, 1)$. Thus $R'_\beta \cap (0, \frac{1}{2})$ would be finitely equivalent with $R'_\beta \cap (0, \frac{1}{2}) \cup (R'_\beta \cap (0, \frac{1}{2})) + \frac{1}{2}$. A general theorem of Lindenbaum and Tarski[9] shows that this is not possible, which completes our proof.

Sierpinski[10] constructed a set k of real numbers of power c whose complement has also power c, and such that if $k \cong k'$ then the power of $k' \cap (R - k)$ [as before R denotes the set of all real numbers] is $< c$. It is easy to see that if we define R_β as in III but remove the restriction $0 \leqq x \leqq 1$. Our set R_β has the required property.

[8] This statement follows from the fact that there exist sets of absolute measure 0 having power c.

[9] Lindenbaum and Tarski.

[10] W. Sierpinski, Fund. Math. Vol. 19, p. 22–28.

We are going to prove that Sierpinski's theorem can not be improved i.e. if m is a power $<c$ there exists a number x_0 such that $k + x_0 \cap R - k$ has power $\geqq m$. Let y_α be any set of real numbers of power m, and suppose that our theorem does not hold. Then both $k + y_\alpha \cap R - k$ and $(R - k) + y_\alpha \cap k$ have power $<m$ for all α. Therefore since $m^2 = m$ it is easy to see that there exists a $z \in k$ and a $w \in R - k$ such that $z + y_\alpha \in k$ and $w + y_\alpha \in R - k$ for all α. But then clearly $x_0 = w - z$ has the required property, which completes our proof.[11]

IV. Let $f(x)$ be a continuous function in the closed interval (0, 1). Denote by E the set for which

$$\overline{\lim_{h \to +0}} \frac{f(x+h) - f(x)}{h} < \infty .$$

Jarnik[12] proved that E is not countable. We are going to give a very simple proof that E is of power c. (It is easy to see that the complement of E is an F_σ, thus from the fact that E is not enumerable it immediately follows that E is of power c.)

Let x_0 be a number <1 for which $\overline{\lim} \frac{f(x+h) - f(x)}{h} < \infty$. We can of course assume that such a number exists. Let $N > \frac{f(1) - f(x_0)}{1 - x_0}$, and consider the set of numbers for which $\frac{f(y) - f(x_0)}{y - x_0} \geqq N$. Consider the greatest such y and denote it by y_N. Clearly $y_N < 1$. Hence evidently

$$\frac{f(y_N + h) - f(y_N)}{h} < N \quad \text{for} \quad h < 0.$$

Thus y_N belongs to E. Also we have $f(y_N) - f(x) = N(y_N - x)$, hence for $N_1 > N_2$, $y_{N_1} < y_{N_2}$. Thus the power of points y_N is c, which completes the proof.

Professor Anthony P. Morse communicated to me the following proof of Jarnik's theorem which he obtained some time ago: Choose k so that if we put $g(x) = f(x) - kx$ we shall have $g(0) > g(1)$. Now take any number c such that $g(0) > c > g(1)$. There clearly exists an x such that $g(x) = 0$. Let x_c be the largest such x. It is easily seen that

$$\overline{\lim_{h \to +0}} \frac{g(x_c + h) - g(x_c)}{h} \leqq 0$$

and hence

$$\overline{\lim_{h \to +0}} \frac{f(x_c + h) - f(x_c)}{a} \leqq k.$$

Thus x_c belongs to E. The power of points x_c is clearly equal to that of the continuum, which completes the proof.

UNIVERSITY OF PENNSYLVANIA

[11] This proof is due to Mr. P. Lax. Oral communication.

[12] Jarnik, Fund. Math. Vol. 23, p. 1-8.

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CORRECTIONS TO TWO OF MY PAPERS

BY P. ERDÖS

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In my paper *"On the divergence properties of the Lagrange interpolation polynomials,"* (Annals of Math. Vol. 42, (1941), p. 309–315) I stated that, if $x_0 = \cos \frac{p}{q}\pi$ (p and q odd), and the fundamental points of the interpolation are the roots of the Tchebicheff polynomial $T_n(x)$, then there exists a continuous function $f(x)$ such that $\lim L_n(f(x_0)) = \infty$.

Dr. Schönberg has pointed out that the proof there given is not correct. There is a trivial error in lemma 1; namely, it is possible that $x_i^{(m)} = x_j^{(n)}$. Nevertheless it is possible to save almost everything, practically without modifying the proof. We prove the following slightly weaker.

THEOREM. *There exists a continuous function $f(x)$ such that if $x_0 = \cos \frac{p}{q}\pi$, where p and q are odd, then* $\lim | L_n(f(x_0)) | = \infty$.

Proof. We need

LEMMA 1. *If $x_j^{(m)} \neq x_j^{(n)}$ then $| x_i^{(m)} - x_j^{(n)} | > \frac{1}{m^3}$ for $m \geqq n$.*

Proof. As in the paper.

Everything is now unchanged until the bottom of page 311. We have there

$$f(x) = \sum_{n=n_0}^{\infty} \epsilon_n \frac{f_n(x)}{\sqrt{\log n}}.$$

where $\epsilon_n = \pm 1$ and will be determined later; the definition of $f_n(x)$ is the same as in the paper.

$L_n(\varphi_2(x_0)) = 0$ still holds (p. 313 top). It suffices to show that, for $r > n$, $f_r(x_k^{(n)}) = 0$. And this is true, for otherwise either

$$x_l^{(r)} = x_k^{(n)},$$

which is impossible since $(2l - 1, r) = 1$, or we have

$$x_l^{(r)} \neq x_k^{(n)} \quad \text{and} \quad | x_l^{(r)} - x_k^{(n)} | < \frac{1}{2^{2r}},$$

which does not hold by lemma 1.

Define now $\epsilon_n = \text{signum } L_n(\varphi_1(x_0))$; then clearly

$$| L_n(f(x_0)) | \leqq \left| L_n\left(\frac{\epsilon_n f_n(x_0)}{\sqrt{\log n}}\right) \right|$$

and the rest of the proof is unchanged.

At present I cannot decide whether a continuous function $f(x)$ exists such that $\lim L_n(f(x_0)) = \infty$, or whether a continuous $f(x)$ exists with $\lim L_n(f(x_0)) = a$, where $a \neq f(x_0)$.

Added in proof. By a more careful analysis, I can now show the following theorem: *Let E be any closed set, then there exists a continuous $f(x)$ such that the limit points of $L_n(f(x_0))$ is precisely the set E. The set E can consist of the point $+\infty$ alone.* This of course is a generalization of the result mentioned before.

In my paper *"On some asymptotic formulas in the theory of factorisatio numerorum"* (Annals of Math. Vol. 42, (1941) p. 989–993) the main theorem is stated incorrectly. The correct statement is as follows:

Let $1 < a_1 < a_2 < \cdots$ be a sequence of integers such that for some ρ, $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho}} = 1$ and $\sum \frac{\log a_i}{a_i^{\rho}}$ converges and not all the a_i's are powers of a_1. Denote by $f(n)$ the number of factorisations of n into the a_i's. We consider order in other words a_1a_2 and $a_2 \cdot a_1$ are different factorisations. Also $f(1) = 1$. Denote $F(n) = \sum_{k=1}^{n} f(k)$. Then we have

$$F(n) = cn^{\rho}(1 + 0(1)).$$

The proof remains entirely unchanged: in fact this theorem is the one really proved in the paper.

It might be of some interest to investigate what happens if the conditions of our theorem are not satisfied. There are three cases: I. $\sum_{i=1}^{\infty} \frac{1}{a_i^k}$ diverges for all k. Then it is easy to see that $\lim \frac{F(n)}{n^k} = \infty$ for all k.

II. For all values of k for which $\sum_{i=1}^{\infty} \frac{1}{a_i^k}$ converges $\sum_{i=1}^{\infty} \frac{1}{a_i^k} < 1$. Clearly, there exists a ρ such that for every ϵ, $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho+\epsilon}}$ converges but $\sum \frac{1}{a_i^{\rho-\epsilon}}$ diverges. We can easily see that $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho}}$ converges and is <1. For if $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho}}$ diverged we would have, for sufficiently small ϵ, $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho+\epsilon}} > 1$; and since, for large k, $\sum_{i=1}^{\infty} \frac{1}{a_i^k} < 1$, there would exist a k_0 such that $\sum_{i=1}^{\infty} \frac{1}{a_i^{k_0}} = 1$—which contradicts the hypothesis.

Now we show that

$$\lim \frac{F(n)}{n^{\rho}} = 0 \tag{1}$$

and

$$\lim \frac{F(n)}{n^{\rho-\epsilon}} = \infty. \tag{2}$$

Suppose (1) does not hold. Write $\sum \frac{1}{a_i^{\rho}} = A < 1$ and $c = \limsup \frac{F(u)}{u^{\rho}}$.

We have

$$F(u) = \sum_{i=1}^{\infty} F\left(\frac{u}{a_i}\right) + 1,$$

so that

$$\frac{F(u)}{u^\rho} \leqq c \sum_{i=1}^{\infty} \frac{1}{a_i^\rho} + o(1) < Ac + o(1)$$

for sufficiently large u. This is possible only if $c = 0$. (2) can be shown by similar arguments.

III. There exists a ρ such that $\sum_{i=1}^{\infty} \frac{1}{a_i^\rho} = 1$, but $\sum_{i=1}^{\infty} \frac{\log a_i}{a_i^\rho}$ diverges. It seems likely that in this case

$$\lim \frac{F(n)}{n^\rho} = 0.$$

But I am only able to prove that

$$\underline{\lim} \frac{F(n)}{n^\rho} = 0. \tag{3}$$

Suppose that (3) is not satisfied. Let the greatest lower bound of $\frac{F(n)}{(n+1)^\rho}$ be c $(c > 0)$. Choose k so large that

$$\sum_{i=1}^{k} \frac{\log a_i}{a_i^\rho} > \frac{2}{\rho c}.$$

Denote by $g(n)$ the number of the factorisations of n as the product of the a_i for $i \leqq k$, and let $G(n) = \sum_{u=1}^{n} g(u)$. Clearly for $m \leqq a_k$, $G(m) = F(m)$. Thus for $m \leqq a_k$

$$\frac{G(m)}{(m+1)^\rho} \geqq c.$$

Next we prove that for all m

$$\frac{G(n)}{(n+1)^{\rho'}} \geqq c \tag{4}$$

where $\sum_{i=1}^{k} \frac{1}{a_i^{\rho'}} = 1$, $(\rho' \leqq \rho)$.

Clearly (4) holds for all $n \leqq a_k$. We prove (4) by induction. Assume it for n: we shall prove it for $n + 1$. We have

$$G(n+1) = \sum_{i=1}^{k} G\left[\frac{n+1}{a_i}\right] + 1 \geqq c \sum_{i=1}^{k} \left(\left[\frac{n+1}{a_i}\right] + 1\right)^{\rho'} + 1.$$

Therefore

$$\frac{G(n+1)}{(n+2)^{\rho'}} \geqq c \sum_{i=1}^{k} \frac{1}{a_i^{\rho'}} = c,$$

which proves (4). Thus $\lim \frac{G(n)}{n^{\rho'}} \geqq c$. (We know from my paper that the limit exists.)

Put $h(s) = \sum_{i=1}^{k} \frac{1}{a_i^s}$; then clearly

$$\frac{1}{2 - h(s)} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

Therefore a simple calculation shows that if $\lim \frac{G(n)}{n^{\rho'}}$ exists the limit equals

$$\left(\rho' \sum_{i=1}^{k} \frac{\log a_i}{a_i^{\rho'}}\right)^{-1} < \frac{\rho c}{2\rho'} \leqq \frac{c}{2},$$

which proves (3). It is easy to construct sequences a_i, with $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho}}, = 1$, $\sum_{i=1}^{\infty} \frac{\log a_i}{a_i^{\rho}} = \infty$ and

$$\lim \frac{F(n)}{n^{\rho}} = 0. \tag{5}$$

But I can not prove that (5) holds for all such sequences a_i.

Professor Hille has given the following result. (Acta Arithmetica Vol. 2, p. 140): Let $p_1 < p_2 < \cdots$ be a sequence of primes, and let $a_1 < a_2 < \cdots$ be the integers composed of the p's. Denote by $f(n)$ the number of factorisations of n into the product of the a's, and let $F(n) = \sum_{m=1}^{n} f(m)$. If $\sum_{i=1}^{\infty} \frac{1}{a_i^{\rho}} = 1$ then $\lim \frac{F(n)}{n^{\rho}} = \left(\rho \sum_{i=1}^{\infty} \frac{\log a_i}{a_i}\right)^{-1}$. His proof (which uses the theorem of Wiener Ikehara) seems to apply only if $\sum_{i=1}^{\infty} \frac{\log a_i}{a_i^{\rho}} < \infty$. If (5) is always true in case iii, Hille's result would follow even if $\sum_{i=1}^{\infty} \frac{\log a_i}{a_i^{\rho}} = \infty$.

Recently I found in the literature a few results, which I proved in my paper "Elementary proof of some asymptotic formulas in the theory of partitions" (Annals of Math. Vol. 43). On p. 447 I prove the following result: Denote by $P_r(n)$ the number of partitions of n into powers of r then $\lim \frac{\log P_r(n)}{(\log n)^2} = \frac{1}{2 \log r}$. This result was proved by Mahler (London Math. Soc. Journal, Vol. 15, p. 123.) Mahlers proof is completely different from mine. He also obtains

$$c_1 \frac{r^{-\frac{1}{2}n(n-1)}(zr)^n}{n!} < P_r(rz) < c_2 \frac{r^{-\frac{1}{2}n(n-1)}(rz)^n}{n!}, \quad \text{where} \quad r^{n-1} n \leqq z < r^n(n+1).$$

On p. 448 I prove the following two results:

I. Let $a_1 < a_2 < \cdots$ be a sequence of integers of positive density α, the a's have common factor 1. Denote by $p(n)$ the number of partition of n into the a's. Then $\log p(n) \sim \pi\sqrt{\frac{2}{3}\alpha n}$.

II. Let $a_1 < a_2 < \cdots$ be a sequence of integers such that every large integer is the sum of different a's. Denote by $P(n)$ the number of partitions of n into different a's. Then $\log P(n) \sim \pi\sqrt{\frac{1}{3}\alpha n}$. Similar results were proved by K. Knopp (Schriften der Königsberger Gel. Ges. Math. und Nat. Klasse, 2 Jahr. Heft. 3 1925). His proofs are quite different from mine and are more complicated.

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ADDENDUM

BY EINAR HILLE

The objection raised by Dr. Erdös to formula (4.3) of my paper "A Problem in 'Factorisatio Numerorum' " is well founded. However, the results on pp. 139–140 are entirely correct if the basis P contains only a finite number of primes p_{i_ν}. When the basis is infinite it is necessary to assume that $\lim_{s\to\sigma_0} \zeta(s; P) > 2$ where $\zeta(s; P) = \prod_{\nu=1}^{\infty} [1 - p_{i_\nu}^{-s}]^{-1}$ and $\sigma_0 = \sigma_0(P)$ is the abscissa of convergence of the infinite product. This assumption implies that the equation $\zeta(s; P) = 2$ has a root $\rho(P)$ which exceeds σ_0. If this assumption is satisfied, formulas (3.8), (3.9), (4.1), (4.3), and (5.1) remain valid. If, instead, $\zeta(\sigma_0; P) = 2$ so that $\rho(P) = \sigma_0$, the Ikehara-Wiener theorem does not apply; the analysis breaks down completely and cannot be saved by assuming that $\zeta'(\sigma_0; P)$ is finite. Though formula (4.3) still makes sense, it is at best unproved. If $\zeta(\sigma_0; P) < 2$, the formula becomes meaningless and it is not enough to replace $\rho(P)$ by σ_0 since Erdös has proved [formula (1) above] that in this case $F(n) = o(n^{\sigma_0})$ while it is not necessarily true that $\zeta'(\sigma_0 ; P)$ is infinite.

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ANALYTIC AND MEROMORPHIC CONTINUATION BY MEANS OF GREEN'S FORMULA

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Poincaré was the first to apply methods of potential theory to the study of analytic functions in several complex variables.[1] Using his method of "balayage" he proved for the entire space that meromorphic functional elements whose difference is holomorphic where they overlap can be merged into one meromorphic function. Later Cousin extended the result to the more general case of a polycylindrical domain, but he used the method of iterating the classical Cauchy integral in one complex variable.

In the present paper we will use a greatly simplified approach to Poincaré's original method, and our main purpose is to show that his theorem is closely connected with the theorem of Hartogs that every analytic function in several complex variables can be continued from the connected boundary of a domain into its whole interior. The connecting link is Green's formula for general harmonic functions and a related version of Cauchy's formula. The first application of this method to Hartogs' theorem is due to Fueter. However Fueter has a very involved approach to the method from the standpoint of quaternions and other Cayley number systems. Our starting point is the ordinary Green's formula, and the specialized results will follow from the fact that an arbitrary differential operator with constant coefficients, being commutative with the Laplacean, is also commutative with Green's integral which is the operational inverse to the Laplacean. This approach will yield old and new results in very general versions and will also permit us to extend Poincaré's theorem to functions on the torus.

CHAPTER I

Stokes' Theorem

1. We will consider a Euclidean space E_n, $n \geqq 3$. If A is a pointset then $\bar{A}$ denotes its closure and $\tilde{A}$ some neighborhood of $\bar{A}$ (no matter how small).

[1] For references to the theorem of Poincaré-Cousin see the article by H. Behnke and P. Tullen, Theorie der Funktionen mehrerer komplexen Veränderlichen, in Ergebnisse der Mathematik, volume III, pp. 64–67; for references to the theorem of Hartogs see the same article, p. 50, and two more recent papers by R. Fueter in Commentarii Mathematici Helvetici, volume 12 (1939), pp. 75–80, and volume 14 (1942), pp. 394–400.

Added in proof, Sept. 1943. Formula (53) of the present paper and a proof of theorem 5 based on it have just been published by Enzo Martinelli, Sopra una dimonstrazione di R. Fueter per una teorema di Hartogs, Comm. Helvetici, 15 (1942-3), submitted Jan. 1943. The present author may be permitted to state that these results have been presented by him in a Princeton graduate course in Winter 1940/41 and were subsequently incorporated, in a Princeton doctorate thesis (June 1941) by Donald C. May, entitled: An integral formula for analytic functions of k variables with some applications.

Virtually all our pointsets will be simplices or finite unions of disjoint simplices. Whenever a symbol like B, C, D will be introduced to denote one or several simplices then the same symbol will also be used to denote the corresponding pointset. If B (or C, D) is an m-dimensional simplex in E_n, then it will be always implicitly understood that B is a topological map of the closure $\bar{S}$ of the rectilinear fundamental simplex S in E_m and the mapping transformation is defined and of class one (with regard to differentiability) in $\bar{S}$. All functions introduced will be likewise of class one unless more restrictive assumptions are explicitly stated. It will be sufficient for our purposes to express Stokes' theorem as the formula

$$\int_B d\varphi\, d\psi_1 \cdots d\psi_{m-1} = \int_C \varphi\, d\psi_1 \cdots d\psi_{m-1}. \tag{1}$$

The scalars $\varphi, \psi_1, \cdots, \psi_{m-1}$ are defined in $\bar{B}$ and C is the set of the $(m-1)$-dimensional faces of B. The simplex B is oriented and the orientation of C is related to the orientation of B by the well known combinatorial rule. By algebraic addition the formula extends readily from a simplex to a (finite) simplicial manifold. In particular, if D is a simplicial domain in E_n with boundary $\bar{B}$, and X_α, $\alpha = 1, \cdots, n$, is a vector field in $\bar{D}$, then

$$\int_D \sum_{\alpha=1}^{n} \frac{\partial X_\alpha}{\partial \xi_\alpha}\, dv_\xi = \int_B \sum_{\alpha=1}^{n} X_\alpha \sigma^\alpha, \tag{2}$$

where

$$dv_\xi = d\xi_1 \cdots d\xi_n \tag{3}$$

$$\sigma^\alpha = (-1)^{\alpha-1} d\xi_1 \cdots d\xi_{\alpha-1} d\xi_{\alpha+1} \cdots d\xi_n\,; \tag{4}$$

or, omitting the symbol of summation with respect to α,

$$\int_D \frac{\partial X_\alpha}{\partial \xi_\alpha}\, dv_\xi = \int_B X_\alpha \sigma^\alpha. \tag{5}$$

The symbols (3), (4) are external differential forms, and the symbol (3) will be the ordinary Euclidean volume element if D is cooriented with E_n; this will be always assumed.

2. We consider m-dimensional simplices $B_1, \cdots, B_s$ and $(m-1)$-dimensional simplices $C_1, \cdots, C_r$, each in a fixed orientation and all disjoint, and we assume that each C_p is a face of one or several B_q and that each $(m-1)$-dimensional face of each B_q occurs among the C_p. We introduce the incidence number ϵ_{pq} which is 0 if C_p is not a face of B_q and otherwise is $+1$ or -1 depending on whether the given orientation of C_p coincides with that of B_q or not.

If we put

$$B = B_1 + \cdots + B_s, \qquad C = C_1 + \cdots + C_r, \tag{6}$$

if the functions $\psi_1, \cdots, \psi_{m-1}$ are scalars in $\tilde{B}$, if $\varphi_q(\xi)$ is given in $\tilde{B}_q$, $q = 1, \cdots, s$, and if we put

$$\varphi^p(\xi) = \sum_{q=1}^{s} \epsilon_{pq} \varphi_q(\xi), \qquad p = 1, \cdots r, \tag{7}$$

then (1) implies

$$\sum_{p=1}^{r} \int_{C_p} \varphi^p \, d\psi_1 \cdots d\psi_{m-1} = \sum_{q=1}^{s} \int_{B_q} d\varphi_q \, d\psi_1 \cdots d\psi_{m-1} . \tag{8}$$

In order to assimilate relation (8) to the original relation (1) we introduce the concept of a *conglomerate function*. We take any functional elements φ_q, $q = 1, \cdots, s$, each defined separately on B_q (or $\tilde{B}_q$) without any interrelation, and we call the whole set a conglomerate function on B (or $\tilde{B}$); we denote it either by φ or more explicitly by $\{\varphi_q\}$. If $\{\varphi_q\}$ is defined on $\tilde{B}$, then the functional elements (7) give rise to a new conglomerate function, the latter defined on $\tilde{C}$. The new function will be called the *saltus* of φ on C. We can now interpret (1) to mean (8) if φ in $d\varphi d\psi_1 \cdots d\psi_{m-1}$ is any conglomerate function and the other φ in $\varphi d\psi_1 \cdots d\psi_{m-1}$ is its saltus. The concept of conglomerate functions unifies the common notions of "discontinuity" and "boundary value." In fact, if B is a simplicial manifold, then on any "internal" C_q which separates simplices B_a, B_b, the saltus of φ is $\pm(\varphi_a - \varphi_b)$ and on any boundary face C_q which borders on only one B_c, the saltus is $\pm\varphi$ itself.

3. We are now going to state a lemma of a special nature. If B and C are defined as before, and φ is a conglomerate function on $\tilde{B}$ then for $m \leqq n - 1$ we will investigate the change of the integral

$$\int_B \varphi \, d\psi_1 \cdots d\psi_m$$

for a small deformation of B. It will suffice to consider a deformation in which only one vertex (and the faces containing it) are involved. We denote this vertex by A_0 and its deformed image by A_0'. Consider all those among the simplices $B_1, B_2, \cdots$ which contain the vertex A_0 and denote them by $B_1, \cdots, B_\mu$. Similarly denote by $C_1, \cdots, C_\lambda$ all faces containing A_0. By B_q', $q = 1, \cdots, \mu$, we denote the deformed image of B_q, by B_p'' we denote the m-dimensional complex which is the combinatorial product of A_0' and C_p, in this order of combination, and finally we denote by S_q the $(m + 1)$-dimensional simplex which is the product of A_0' and B_q. We now claim that the difference

$$\sum_{q=0}^{\mu} \int_{B_q} \varphi_q \, d\psi_1 \cdots d\psi_m - \sum_{q=0}^{\mu} \int_{B_q'} \varphi_q \, d\psi_1 \cdots d\psi_m \tag{9}$$

has the value

$$\sum_{q=0}^{\mu} \int_{S_q} d\varphi_q \, d\psi_1 \cdots d\psi_m + \sum_{p=0}^{\lambda} \int_{B_p''} \varphi^p \, d\psi_1 \cdots d\psi_m , \tag{10}$$

where $\{\varphi^p\}$ on $\tilde{C}$ is the saltus of $\{\varphi_q\}$ on $\tilde{B}$.

The proof consists in a simple combinatorial analysis. Consider any fixed simplex B_1 with vertex A_0 and write it combinatorially as $A_0A_1A_2 \cdots A_m$ where $A_1, \cdots, A_m$ are other vertices. We now introduce the $(m+1)$-dimensional simplex S_1, namely the simplex

$$A_0'A_0A_1 \cdots A_m. \tag{11}$$

Its m-dimensional faces in natural orientation are the sum of

$$A_0A_1A_2 \cdots A_m - A_0'A_1A_2 \cdots A_m \tag{12}$$

and

$$A_0'A_0A_2 \cdots A_m - A_0'A_0A_1A_3 \cdots A_m + \cdots \pm A_0'A_0A_1 \cdots A_{m-1}. \tag{13}$$

Now, (12) is simply $B_1 - B_1'$. As for (13), its first term is the product of A_0' with $A_0A_2 \cdots A_m$. The latter is the negative of $\epsilon_{1p}C_p$, for some p, where ϵ_{1p} is our previous incidence coefficient. Thus the first term in (13) is $-\epsilon_{1p}B_p''$ for some p. The same holds for any other term in (13). It is always $-\epsilon_{1p}B_p''$ for some p, the sign being always negative whether it was originally positive or negative in (13). The equality of (9) and (10) follows now by applying Stokes' formula to S_1 and its boundary, and to any other S_q and its boundary, and then adding up over those S_q which correspond to all those B_q which contain the given vertex A_0.

CHAPTER II

Analytic Functions of Real Variables

4. Green's formula. The most familiar general version of Green's formula is

$$\gamma_n \underset{0}{f}(x) = \int_B \left(f \frac{\partial G}{\partial \xi_\alpha} - G \frac{\partial f}{\partial \xi_\alpha} \right) \sigma^\alpha - \int_D (f \cdot \Delta G - G \cdot \Delta f)\, dv_\xi, \tag{14}$$

and we are adding the kindred formula

$$\gamma_n \underset{0}{f}(x) = \int_B f \frac{\partial G}{\partial \xi_\alpha} \sigma^\alpha - \int_D \left(f \cdot \Delta G + \frac{\partial f}{\partial \xi_\alpha} \frac{\partial G}{\partial \xi_\alpha} \right) dv_\xi. \tag{15}$$

The function G is

$$\frac{1}{2-n} \left(\sum_\alpha (\xi_\alpha - x_\alpha)^2 \right)^{\frac{2-n}{2}} + H(\xi, x), \tag{16}$$

where $H(\xi, x)$ is analytic in $E_n \times E_n$; the function f is a function on $\bar{D}$, where D is a simplicial domain with boundary $\bar{B}$; the symbol Δ denotes the Laplacean

$$\Delta = \frac{\partial^2}{\partial \xi_1^2} + \cdots + \frac{\partial^2}{\partial \xi_n^2}; \tag{17}$$

the lower value 0 occurs when x is a point of $E_n - \bar{D}$ whereas the upper value $\gamma_n f(x)$ occurs when x is a point of D; and γ_n is the constant

$$\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}, \tag{18}$$

which constant is the $(n-1)$-dimensional volume of the sphere $\xi_1^2 + \cdots + \xi_n^2 = 1$. For x outside $\bar{D}$ our formulas are direct consequences of (5) if $X_\alpha = f\frac{\partial G}{\partial \xi_\alpha} - G\frac{\partial f}{\partial \xi_\alpha}$, or $X_\alpha = f\frac{\partial G}{\partial \xi_\alpha}$, and for x in D they are obtained from the same values X_α by first excising a small sphere with center at x and then letting the radius of the sphere tend to 0.

If $H(\xi, x) = 0$ in (16), that is

$$G(\xi, x) \equiv \frac{1}{2-n}\left(\sum_\alpha (\xi_\alpha - x_\alpha)^2\right)^{\frac{2-n}{2}} \tag{19}$$

then G is harmonic, that is a solution of $\Delta G = 0$. If in addition $f(\xi)$ is also harmonic then (14) reduces to

$$\int_B \left(f\frac{\partial G}{\partial \xi_\alpha} - G\frac{\partial f}{\partial \xi_\alpha}\right)\sigma^\alpha = \gamma_n f(x) \quad \text{or } 0. \tag{20}$$

Until further notice, the symbol G will always denote the Newtonian kernel (19).

5. Green's integral. We now consider a fixed set of simplices (6) and we specify that the dimension of B shall be $n-1$. We will call a conglomerate function $\{f_q\}$ on $\tilde{B}$ harmonic if each f_q is harmonic on $\tilde{B}_q$. With such a conglomerate harmonic function $f(\xi)$ we now set up the integral

$$F(x) = \frac{1}{\gamma_n}\int_B \left(f\frac{\partial G}{\partial \xi_\alpha} - G\frac{\partial f}{\partial \xi_\alpha}\right)\sigma^\alpha. \tag{21}$$

The complement $E_n - \bar{B}$ decomposes into a finite number of domains $D_1, \cdots D_t, D_\infty$. The domain D_∞ is unbounded, the others are bounded. The bounded ones may be absent; this will happen if $\bar{B}$ does not decompose the space E_n. The union $D_1 + \cdots + D_t + D_\infty$ will be denoted by D, and we will treat D as if it were a union of simplices. Integral (21) defines a harmonic function $F_\tau(x)$ in each D_τ, $\tau = 1, \cdots, t, \infty$. In this sense, $\{F_\tau(x)\}$ is a harmonic conglomerate function on D.

THEOREM 1. *If $f(\xi)$ is a harmonic conglomerate function on $\tilde{B}$, then each $F_\tau(x)$ can be continued analytically across each B_q bordering on D_τ, and the saltus of $F(x)$ on B_q is $f_q(x)$.*

For the proof it may be assumed that B is just one simplex. We deform B into another simplex B' with the same boundary as B in such a way that $B' - B$ shall be part of the boundary of a simplicial domain D. The "side" of B on

which D is situated will be termed "positive," the other "negative." If x lies on the negative side of B it lies outside $\bar{D}$ and thus by (20) we have

$$F(x) = \frac{1}{\gamma_n}\int_{B'}\left(f\frac{\partial G}{\partial \xi_\alpha} - G\frac{\partial f}{\partial \xi_\alpha}\right)\sigma^\alpha. \tag{22}$$

Integral (22) however can be continued across B into D. Denoting this continuation by $F^+(x)$ we obviously have for x in D the relation

$$F^+(x) - F(x) = \frac{1}{\gamma_n}\int_{B'-B}\left(f\frac{\partial G}{\partial \xi_\alpha} - G\frac{\partial f}{\partial \xi_\alpha}\right)\sigma^\alpha. \tag{23}$$

However, again by (20), this is $f(x)$, and thus the theorem is proven.

THEOREM 2. *If also the saltus of $f(\xi)$ vanishes identically on $\tilde{C}$, then each F_τ can be continued into $\tilde{D}$.*

More generally, if a small deformation of a vertex A_0 of B enlarges a component D_τ into a domain D'_τ, and if the saltus of $f(\xi)$ vanishes on all those $\tilde{C}_p$ which contain the vertex A_0, then $F_\tau(x)$ can be continued from D_τ into D'_τ.

If we introduce the symbols B_q, B'_q, B''_p and S_q as in section 3, then by section 3 the expression

$$\sum_{q=0}^{\mu}\int_{B_q}\left(f_q\frac{\partial G}{\partial \xi_\alpha} - G\frac{\partial f_q}{\partial \xi_\alpha}\right)\sigma^\alpha - \sum_{q=0}^{\mu}\int_{B'_q}\left(f_q\frac{\partial G}{\partial \xi_\alpha} - G\frac{\partial f_q}{\partial \xi_\alpha}\right)\sigma^\alpha \tag{24}$$

is the sum of

$$\sum_{q=0}^{\mu}\int_{S_q}(f_q\cdot\Delta G - G\cdot\Delta f_q)\,dv_\xi \tag{25}$$

and

$$\sum_{p=0}^{\lambda}\int_{B''_p}\left(f^p\frac{\partial G}{\partial \xi_\alpha} - G\frac{\partial f^p}{\partial \xi_\alpha}\right)\sigma^\alpha. \tag{26}$$

Now, (24) is $\gamma_n(F(x) - F'(x))$ where the integral $F'(x)$ is what will become of $F(x)$ if B is replaced by its deformation B'. The sum (25) is naturally 0, and (26) is 0 if $f^p = 0$ on B''_p, that is if the saltus of f vanishes on $\tilde{C}_p$. Thus $F(x) = F'(x)$, but $F'(x)$ exists in D'_τ, and this proves our assertion.

6. Analytic continuation. We are now viewing the expression (21) as an operation

$$F(x) = Lf(\xi) \tag{27}$$

which transform a function in $\tilde{B}$ into a harmonic function in D. We claim that this operation is commutative with partial derivation.

THEOREM 3. *If $f(\xi)$ is harmonic in $\tilde{B}$ then*

$$L\frac{\partial f}{\partial \xi_\beta} = \frac{\partial}{\partial x_\beta}F(x), \tag{28}$$

provided the saltus of $f(\xi)$ and of $\dfrac{\partial f}{\partial \xi_1}, \cdots, \dfrac{\partial f}{\partial \xi_n}$ each vanishes on C.

More generally if the saltus of $f(\xi)$ and its partial derivatives of order $\leqq N$ vanishes on C, then

$$L\Delta f = \Delta L f \tag{29}$$

where

$$\Delta f \equiv \sum_{\nu_1+\cdots+\nu_n} a_{\nu_1 \cdots \nu_n} \frac{\partial^{\nu_1+\cdots+\nu_n} f}{\partial \xi_1^{\nu_1} \cdots \partial \xi_n^{\nu_n}} \tag{30}$$

with constant coefficients a.

REMARK. It should be noted that in Theorem 2, the saltus of f has to vanish identically on the open set $\bar{C}$, whereas in Theorem 3 the saltuses have only to satisfy the *boundary condition of vanishing on the $(n-2)$-dimensional set C.*

PROOF. Obviously (29) follows from (28) by induction on N. In addition to the symbols σ^α as defined by (4), we now introduce symbols $\sigma^{\alpha\beta}$ in the following way. For $\alpha < \beta$ we put

$$\sigma^{\alpha\beta} = (-1)^{\alpha+\beta-1} d\xi_1 \cdots d\xi_{\alpha-1} d\xi_{\alpha+1} \cdots d\xi_{\beta-1} d\xi_{\beta+1} \cdots d\xi_n;$$

for $\alpha = \beta$ we put $\sigma^{\alpha\beta} = 0$; and for $\alpha > \beta$ we put $\sigma^{\alpha\beta} = -\sigma^{\beta\alpha}$. The basis for our proof is the identity

$$\sum_{\alpha=1}^{n} \left(f \frac{\partial^2 G}{\partial \xi_\beta \partial \xi_\alpha} + \frac{\partial f}{\partial \xi_\beta} \frac{\partial G}{\partial \xi_\alpha} \right) \sigma^\alpha = \left(f \cdot \Delta G + \sum_{\alpha=1}^{n} \frac{\partial f}{\partial \xi_\alpha} \frac{\partial G}{\partial \xi_\alpha} \right) \sigma^\beta - \sum_{\alpha=1}^{n} d\left(f \frac{\partial G}{\partial \xi_\alpha} \right) \sigma^{\beta\alpha}. \tag{31}$$

From this we deduce that the quantity

$$\sum_{\alpha=1}^{n} \left(\frac{\partial f}{\partial \xi_\beta} \frac{\partial G}{\partial \xi_\alpha} - G \frac{\partial^2 f}{\partial \xi_\beta \partial \xi_\alpha} + f \frac{\partial^2 G}{\partial \xi_\beta \partial \xi_\alpha} - \frac{\partial G}{\partial \xi_\beta} \frac{\partial f}{\partial \xi_\alpha} \right) \sigma^\alpha \tag{32}$$

equals the quantity

$$(f \cdot \Delta G - G \cdot \Delta f) \sigma^\beta - \sum_{\alpha=1}^{n} d\varphi_\alpha \cdot \sigma^{\beta\alpha}, \tag{33}$$

where

$$\varphi_\alpha = f \frac{\partial G}{\partial \xi_\alpha} - G \frac{\partial f}{\partial \xi_\alpha}. \tag{34}$$

Now, since $\frac{\partial G}{\partial x_\beta} = -\frac{\partial G}{\partial \xi_\beta}$, the difference

$$\gamma_n \left(L \frac{\partial f}{\partial \xi_\beta} - \frac{\partial}{\partial x_\beta} F \right)$$

is the integral of (32) over B. By the equality of (32) and (33) this is

$$-\int_B \sum_{\alpha=1}^{n} d\varphi_\alpha \cdot \sigma^{\beta\alpha}$$

and by section 2 this will vanish if the saltus of each φ_α vanishes on C. By (34), it is enough to assume that $f, \frac{\partial f}{\partial \xi_1}, \cdots, \frac{\partial f}{\partial \xi_n}$ has vanishing saltus each.

We note a natural consequence of Theorem 3.

COROLLARY. *If the conglomerate function $f(\xi)$ is harmonic and the solution of $\Lambda f = 0$ and if the saltuses of f and of its partial derivatives of order $\leqq N$ vanish on C then $F(x)$ is likewise harmonic and a solution of $\Lambda F = 0$.*

We will say that a differential operator ΛF of type (30) has the *uniqueness property* if any analytic solution of $\Lambda F = 0$ which is defined in the complement of a bounded domain and vanishes at infinity is identically 0. Such an operator is for instance

$$\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2} \tag{35}$$

for $m < n$. In fact, if a function $F(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n)$ is defined in the complement of a bounded domain in E_n then there exists an $(n - m)$-dimensional neighborhood of values $(x_{m+1}^0, \cdots, x_n^0)$, such that for each of those values, $F(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n)$ is defined in

$$-\infty < x_\mu < \infty, \qquad \mu = 1, \cdots, m.$$

By our assumption it is a harmonic function in the entire $(x_1, \cdots, x_m)$-space and vanishes at infinity. Hence it is identically 0 in the latter variables; but being analytic in all n variables, it vanishes identically in all variables.

THEOREM 4. *If a conglomerate function $f(\xi)$ is harmonic in $\tilde{B}$ and the solution of an equation $\Lambda f = 0$ with uniqueness property; and if the saltuses of $f(\xi)$ and of its partial derivatives of order $\leqq N$ vanish on C then the following propositions hold.*

(i) *If the component D_∞ of $E_n - \bar{B}$ borders on a simplex B_p from both sides then $f_p(x) = 0$.*

(ii) *In particular, if B does not decompose the space then $f(x) = 0$.*

(iii) *If a simplex B_p separates D_∞ from D_τ then $f_p(x)$ can be continued into all of D_τ.*

(iv) *In particular, if $\bar{B}$ is the connected boundary of a bounded domain D then $f(x)$ can be continued (uniquely) into all of D.*

PROOF. *Ad* (i) *and* (ii). By the hypotheses of the theorem, $F_\infty(x)$ vanishes. But, by Theorem 3, in crossing over from D_∞ into D_∞ via B_p, $F(x)$ jumps by $\pm f_p(x)$. Hence $f_p(x) = 0$.

Ad (iii) *and* (iv). On B_p, $F(x)$ has the saltus $f_p(x)$. Therefore $F_\tau(x) - F_\infty(x) = \pm f_p(x)$. However, $F_\infty(x) = 0$. Thus, near B_p, $F_\tau(x)$ coincides with $\pm f_p(x)$. However, $F_\tau(x)$ exists in all of $\tilde{D}_\tau$. Therefore $f_p(x)$ exists in all of D_τ.

Now consider the space E_{2k}, $k \geqq 2$, of the real variables $x_1, y_1, \cdots, x_k, y_k$ and put $z_\alpha = x_\alpha + iy_\alpha$. If $f(z_1, \cdots, z_k)$ is an analytic function of the complex variables $z_1, \cdots, z_k$, then in particular its real and imaginary parts each satisfy the system of equations

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} = 0, \cdots, \qquad \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} = 0. \tag{36}$$

Each of these equations has the uniqueness property. Also a simultaneous solution of them is harmonic in $(x_1, y_1, \cdots, x_k, y_k)$. Hence we obtain the following special case of theorem 4, (iv).

THEOREM 5. *If an analytic function of several complex variables is defined in the connected boundary of a bounded domain, then it can be continued uniquely into all of the domain.*

CHAPTER III

Analytic Functions of Complex Variables

7. Complex space. We will now consider systematically the space E_{2k} of k complex variables $\zeta_\alpha = \xi_\alpha + i\eta_\alpha$. All occurring functions will be likewise complex-valued. In particular a harmonic function will be one whose real and imaginary part are each harmonic. Replacing the pair of real variables ξ, η by the conjugate complex quantities $\zeta = \xi + i\eta$, $\bar{\zeta} = \xi - i\eta$ we have

$$d\bar{\zeta}d\zeta = 2id\xi d\eta \tag{37}$$

and therefore

$$d\bar{\zeta}_1 d\zeta_1 \cdots d\bar{\zeta}_k d\zeta_k = (2i)^k d\xi_1 d\eta_1 \cdots d\xi_k d\eta_k . \tag{38}$$

If any two of the $2k$ real factors on the right side of (38) are interchanged, the symbol changes its algebraic sign. However, if any two blocks of terms $d\bar{\zeta}_\alpha d\zeta_\alpha$, $d\bar{\zeta}_\beta d\zeta_\beta$ are interchanged the sign is not altered. It will be worth while to introduce the differential form

$$\omega = d\bar{\zeta}_1 d\zeta_1 + \cdots + d\bar{\zeta}_k d\zeta_k . \tag{39}$$

It is an "algebraic" area element, and *not* the scalar square of a length element. As for its (external) powers $\omega^2 = \omega \cdot \omega$, $\omega^3 = \omega^2 \cdot \omega$, $\cdots$, it can be easily seen that

$$\omega^k = k! d\bar{\zeta}_1 d\zeta_1 \cdots d\bar{\zeta}_k d\zeta_k \tag{40}$$

and that, for each α,

$$d\bar{\zeta}_\alpha d\zeta_\alpha \omega^{k-1} = \frac{1}{k}\,\omega^k . \tag{41}$$

As for the theorem of Stokes, we note that (1) remains valid for all occurring functions being complex-valued. In particular, if D is a domain with boundary B, then

$$\frac{1}{k}\int_D \frac{\partial X_\alpha}{\partial \bar{\zeta}_\alpha}\,\omega^k = \int_B X_\alpha d\zeta_\alpha \omega^{k-1} \tag{42}$$

and

$$-\frac{1}{k}\int_D \frac{\partial Y_\alpha}{\partial \zeta_\alpha}\,\omega^k = \int_B Y_\alpha d\bar{\zeta}_\alpha \omega^{k-1} . \tag{43}$$

We now put

$$V = 2G \tag{44}$$

where G is the function (16). Thus in the special case (19) we have

$$V = \frac{1}{1-k}\left(\sum_{\alpha}(\zeta_\alpha - z_\alpha)(\bar{\xi}_\alpha - \bar{z}_\alpha)\right)^{1-k} \tag{45}$$

$$\frac{\partial V}{\partial \zeta_\alpha} = \frac{\bar{\xi}_\alpha - z_\alpha}{\left(\sum_{\beta} |\zeta_\beta - z_\beta|^2\right)^k}. \tag{46}$$

In the general case (16), on putting

$$X_\alpha = f\frac{\partial V}{\partial \zeta_\alpha}, \qquad Y_\alpha = V\frac{\partial f}{\partial \bar{\xi}_\alpha}$$

we obtain, as analogues to (14) and (15) the formulas

$$\begin{aligned}(2\pi i)^k {}_0 f(z,\bar{z}) = {} & \int_B \left(f\frac{\partial V}{\partial \zeta_\alpha}\,d\zeta_\alpha + V\frac{\partial f}{\partial \bar{\xi}_\alpha}\,d\bar{\xi}_\alpha\right)\omega^{k-1} \\ & -\frac{1}{k}\int_D \left(f\frac{\partial^2 V}{\partial \zeta_\alpha \partial \bar{\xi}_\alpha} - V\frac{\partial^2 f}{\partial \zeta_\alpha \partial \bar{\xi}_\alpha}\right)\omega^k\end{aligned} \tag{47}$$

and

$$(2\pi i)^k {}_0 f(z,\bar{z}) = \int_B f\frac{\partial V}{\partial \zeta_\alpha}\,d\zeta_\alpha \omega^{k-1} - \frac{1}{k}\int_D \left(f\frac{\partial^2 V}{\partial \zeta_\alpha \partial \bar{\xi}_\alpha} + \frac{\partial f}{\partial \bar{\xi}_\alpha}\frac{\partial V}{\partial \zeta_\alpha}\right)\omega^k. \tag{48}$$

Since

$$4\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}$$

we have

$$4\sum_{\alpha=1}^{k}\frac{\partial^2}{\partial \zeta_\alpha \partial \bar{\zeta}_\alpha} = \sum_{\alpha=1}^{k}\left(\frac{\partial^2}{\partial \xi_\alpha^2} + \frac{\partial^2}{\partial \eta_\alpha^2}\right) \equiv \Delta. \tag{49}$$

Thus, since V is harmonic, if we also assume that f is harmonic, then (46) leads to

$$\int_B \left(f\frac{\partial V}{\partial \zeta_\alpha}\,d\zeta_\alpha + V\frac{\partial f}{\partial \bar{\xi}_\alpha}\,d\bar{\xi}_\alpha\right)\omega^{k-1} = (2\pi i)^k f(z,\bar{z}) \quad or \quad 0, \tag{50}$$

which is the analogue to (20). It should be observed that (50) is genuinely different from (20). All that can be said is that for real f, relation (50) is "the real part" of relation (20). The difference will be significant for general point-sets B, and will in fact lead to an improvement of Theorem 4.

8. Cauchy's formula. Now, if in particular f is an analytic function in $z_1, \cdots, z_k$ alone, thus

$$\frac{\partial f}{\partial \bar{z}_\alpha} = 0, \qquad \alpha = 1, \cdots, k, \tag{51}$$

(which assumption in particular implies that f is harmonic) then (50) reduces to the very simple formula

$$f(z) \text{ or } 0 = \frac{1}{(2\pi i)^k} \int_B f \frac{\partial V}{\partial \zeta_\alpha} d\zeta_\alpha \omega^{k-1}. \tag{52}$$

If we write the formula in full we obtain

THEOREM 6. *If $f(z)$ is analytic in $\bar{D}$ then the formula*

$$f(z) = \frac{1}{(2\pi i)^k} \int_B \frac{f(\zeta)(\bar{\zeta}_\alpha - \bar{z}_\alpha)\, d\zeta_\alpha \omega^{k-1}}{\left(\sum_\beta (\zeta_\beta - z_\beta)(\bar{\zeta}_\beta - \bar{z}_\beta)\right)^k} \tag{53}$$

holds for z in D.

This is a generalization of Cauchy's formula in one variable, however with the peculiarity that in the cases $k \geqq 2$ we are considering, the integrand contains $\bar{z}_\beta$ explicitly. Thus to start with it is not at all evident that the integral will be independent of $\bar{z}_\beta$ let alone be $f(z)$.

Now take a point $(z_1, \cdots, z_k)$ in D and replace in the integral (53) the conjugate point $(\bar{z}_1, \cdots, \bar{z}_k)$ by a point $(z_1', \cdots, z_k')$ in the vicinity of the conjugate point. The right side of (53) will then be a function

$$F(z_1, \cdots, z_k; z_1', \cdots, z_k')$$

in $2k$ complex variables. By Theorem 6, the function

$$F(z_1, \cdots, z_k; z_1', \cdots, z_k') - f(z_1, \cdots, z_k) \tag{54}$$

vanishes on the manifold $z_1' = \bar{z}_1, \cdots, z_k' = \bar{z}_k$. This implies that (54) vanishes identically, that is

$$f(z_1, \cdots, z_k) \equiv F(z_1, \cdots, z_k; z_1', \cdots, z_k').$$

Now assume that D contains the origin, and let $z_1', \cdots, z_k'$ be the origin. Then, at least for points z near the origin we have

$$f(z) = \frac{1}{(2\pi i)^k} \int_B \frac{f(\zeta)\bar{\zeta}_\alpha\, d\zeta_\alpha \omega^{k-1}}{[(\zeta_1 - z_1)\bar{\zeta}_1 + \cdots + (\zeta_k - z_k)\bar{\zeta}_k]^k}. \tag{55}$$

This formula is much closer to Cauchy's one-variable formula than (53) is, however the integral (55) does not necessarily converge in all of D. The domain of convergence can be described geometrically, but the description would be of no consequence in the present context.

9. Cauchy's integral. As an analogue to (21) we now introduce the integral

$$F(z, \bar{z}) = \frac{1}{(2\pi i)^k} \int_B f(\zeta) \frac{\partial V}{\partial \zeta_\alpha} d\zeta_\alpha \omega^{k-1} \tag{56}$$

for an analytic conglomerate function $f(\zeta_1, \cdots, \zeta_k)$ in $\tilde{B}$. We first of all have the following analogue to Theorems 1 and 2.

THEOREM 7. *The saltus of the conglomerate function* (56) *on* B *is* $f(z)$.

Also if the saltus of $f(\zeta)$ *vanishes identically on all* $\tilde{C}_p$ *containing a vertex* A_0 *of* B, *then a small deformation of* A_0 *leads to analytic continuation of* $F_\tau(z, \bar{z})$ *from* D_τ *into* D'_τ.

The following theorem is an analogue to Theorem 3 with

$$\Lambda f \equiv \frac{\partial f}{\partial \bar{z}_\beta} = \frac{\partial f}{\partial x_\beta} + i \frac{\partial f}{\partial y_\beta}, \tag{57}$$

but it is a great improvement on Theorem 3, insofar as it does not involve a boundary condition on the *derivatives* of f.

THEOREM 8. *If the saltus of* $f(\zeta)$ *vanishes on* C, *then* F *is analytic in* $z_1, \cdots, z_k$, *that is*, $\dfrac{\partial F}{\partial \bar{z}_\beta} = 0, \beta = 1, \cdots, k.$

Also, in D_∞ *the functional element* $F_\infty(z)$ *is identically* 0.

For use in the proof, we introduce the symbol $s^{\bar{\alpha}}$ which is the differential form arising from

$$d\bar{\zeta}_1 d\zeta_1 \cdots d\bar{\zeta}_k d\zeta_k \tag{58}$$

by dropping $d\bar{\zeta}_\alpha$ and by $s^{\bar{\alpha}\bar{\beta}}$ we denote the symbol arising from (28) by dropping both $d\bar{\zeta}_\alpha$ and $d\bar{\zeta}_\beta$ if $\alpha \neq \beta$, and the symbol 0 if $\alpha = \beta$. Now

$$\frac{(2\pi i)^k}{(k-1)!} \frac{\partial F}{\partial \bar{z}_\beta} = \int_B - \frac{1}{(k-1)!} \sum_{\alpha=1}^{k} f \frac{\partial^2 V}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta} d\zeta_\alpha \omega^{k-1}, \tag{59}$$

but the integral is, identically in f, V,

$$f \cdot \Delta V \cdot s^{\bar{\beta}} + \sum_{\alpha=1}^{k} d\left(f \frac{\partial V}{\partial \zeta_\alpha}\right) s^{\bar{\alpha}\bar{\beta}} - \sum_{\alpha=1}^{k} \frac{\partial V}{\partial \zeta_\alpha} df s^{\bar{\alpha}\bar{\beta}}. \tag{60}$$

In our case we have $\Delta V = 0$, and we also have df. $s^{\bar{\alpha}\bar{\beta}} = 0$ since f is independent of both $\bar{\zeta}_\alpha$ and $\bar{\zeta}_\beta$. Therefore, by Stokes' theorem, (59) is

$$\int_C f \frac{\partial V}{\partial \zeta_\alpha} s^{\bar{\alpha}\bar{\beta}}$$

and thus (59) vanishes if the saltus of f vanishes on C. The second half of the theorem has been proven in section 6.

Finally we prove the following analogue of Theorem 4.

THEOREM 9. *If the conglomerate function* $f(\zeta_1, \cdots, \zeta_k)$ *is analytic in* B *and if its saltus vanishes on* C, *then the following propositions hold:*

(i) *If the component* D_∞ *borders on a simplex* B_p *from both sides then* $f_p(\zeta) = 0$.

(ii) *If, in particular,* $\tilde{B}$ *does not decompose* E_n, *then* $f(\zeta) = 0$.

(iii) *If a simplex* B_p *separates* D_∞ *from* D_τ *then* $f_p(\zeta)$ *can be continued into all of* D_τ.

As a special conclusion we observe that if the set of all zeroes of an analytic function $f(\zeta_1, \cdots, \zeta_k)$ contains a bounding cycle of dimension $2k - 2$ then the function vanishes identically.

As a counterpart to this property we observe that if $f(\zeta_1, \cdots, \zeta_k)$ is $\neq 0$ everywhere on the boundary $\bar{B}$ of a domain D then $f \neq 0$ everywhere in D. In fact f^{-1} is analytic in a neighborhood of B and therefore it can be continued into all of D.

CHAPTER IV

Integrability Conditions

10. We are interested in the system of equations

$$\begin{aligned} &\frac{\partial f}{\partial \bar{z}_\alpha} = f_\alpha(z, \bar{z}), && \alpha = 1, \cdots, k, \\ &\frac{\partial f_\alpha}{\partial \bar{z}_\beta} = \frac{\partial f_\beta}{\partial \bar{z}_\alpha}, && \alpha, \beta = 1, \cdots, k, \end{aligned} \tag{61}$$

in any domain D in E_{2k}. The functions f_α are given and the function f is a desired solution. The system is in some respects similar to the classical system

$$\begin{aligned} &\frac{\partial f}{\partial x_\alpha} = f_\alpha(x), && \alpha = 1, \cdots, n, \\ &\frac{\partial f_\alpha}{\partial x_\beta} = \frac{\partial f_\beta}{\partial x_\alpha}, && \alpha, \beta = 1, \cdots, n, \end{aligned} \tag{62}$$

in a domain D in E_n. Assuming all functions f_α to be analytic in x or $(z, \bar{z})$ respectively then either system has an analytic solution in a neighborhood of a point. For (62) this follows easily if all $f_\alpha(x)$ are expanded in power series. Also if the functions $f_\alpha(x)$ contains additional analytic parameters then so does the solution f. Hence we may conclude the existence of local solutions of (61) by viewing $\bar{z}_1, \cdots, \bar{z}_k$ as independent variables and $z_1, \cdots, z_k$ as parameters.

As for solutions in the large, we will first prove a theorem which will apply to either system (61) or (62). The theorem will not be needed, but the proof will be based on an application of Green's formula which apparently has not been noticed before. After that we will prove another theorem for the system (61). It will be the exact counterpart within the scope of the general Green's integral to Cousin's method as based on the classical Cauchy's integral.

THEOREM 10. *If D is a simplicial domain with boundary $\bar{B}$, if $\{f_\alpha\}$ is given in D and if either system* (61) *or* (62) *has a solution in $\bar{B}$, then there also exists a solution in D.*

We will treat only system (62) for system (61) the proof is analogous. With the given functions f_α in $\bar{D}$ and the solution f in $\bar{B}$ we set up the integral

$$\gamma_n F(x) = \int_B f \frac{\partial G}{\partial \xi_\alpha} \sigma^\alpha - \int_D f_\alpha \frac{\partial G}{\partial \xi_\alpha} \, dV_\xi .$$

By formula (15), if f exists in all of $\bar{D}$, then $F(x) = f(x)$. Now take a fixed point x^0 in D and a sufficiently small neighborhood D' with boundary $\bar{B}'$. If it is sufficiently small, then there exists a function $g(x)$ with $f_\alpha = \dfrac{\partial g}{\partial x_\alpha}$ in D' such that, for x in D',

$$\gamma_n g(x) = \int_{B'} f \frac{\partial G}{\partial \xi_\alpha} \sigma^\alpha - \int_{D'} f_\alpha \frac{\partial G}{\partial \xi_\alpha} dV_\xi .$$

The reader will now easily see that our theorem will follow from the following

LEMMA. *If the boundary B of a domain D consists of two disjoint parts, $B = B' + B''$, if f in $\bar{B}'$ and g in $\bar{B}''$ are each a solution of* (62), *then for x in $E_n - \bar{D}$, the partial derivatives of the function*

$$\int_{B'} f \frac{\partial G}{\partial \xi_\alpha} \sigma^\alpha + \int_{B''} g \frac{\partial G}{\partial \xi_\alpha} \sigma^\alpha - \int_D f_\alpha \frac{\partial G}{\partial \xi_\alpha} dV_\xi$$

vanish identically.

In order to prove the lemma we note that the partial derivative of the given expression with respect to x_β, for x outside $\bar{D}$, is

$$-\int_{B'} f \frac{\partial^2 G}{\partial \xi_\alpha \partial \xi_\beta} \sigma^\alpha - \int_{B''} g \frac{\partial^2 G}{\partial \xi_\alpha \partial \xi_\beta} \sigma^\alpha + \int_D f_\alpha \frac{\partial^2 G}{\partial \xi_\alpha \partial \xi_\beta} dV_\xi .$$

By (31) this is

$$\int_B f_\beta \frac{\partial G}{\partial \xi_\alpha} \sigma^\alpha - \int_B f_\alpha \frac{\partial G}{\partial \xi_\alpha} \sigma^\beta + \int_D f_\alpha \frac{\partial^2 G}{\partial \xi_\alpha \partial \xi_\beta} dV_\xi .$$

On applying Stokes' theorem to the integrals over B, this turns out to be

$$\int_D \left(\frac{\partial f_\beta}{\partial \xi_\alpha} - \frac{\partial f_\alpha}{\partial \xi_\beta} \right) \frac{\partial G}{\partial \xi_\alpha} dV_\xi$$

and this in its turn is obviously 0.

11. An existence theorem. We will now prove a genuine existence theorem.

THEOREM 11. *The system* (61) *has a solution if the domain D is a general polycylinder, that is the topological product of a domain in each of the variables $z_1, \cdots, z_k$ separately.*

The theorem will follow from a lemma on functions in the plane of one complex variable $z = x + iy$.

If $\varphi(z, \bar{z})$ is analytic in $\bar{D}$, then the integral

$$F(z, z) = \frac{-1}{2\pi i} \int_D \frac{\varphi(\zeta, \bar{\zeta})\, d\bar{\zeta}\, d\zeta}{\zeta - z} \tag{63}$$

is analytic in z, $\bar{z}$ and $\dfrac{\partial F}{\partial \bar{z}} = \varphi(z, \bar{z})$. Also if φ depends analytically on additional parameters, then so does F.

For z outside $\bar{D}$ the integral (63) is obviously analytic in z and other parameters and independent of $\bar{z}$. Hence, in order to prove that the stated properties of $F(z, z)$ hold in the neighborhood of a point z^0 in D we may assume that D is an arbitrarily small neighborhood of z^0. If the neighborhood is sufficiently small then there exists a function $\Phi(z, \bar{z})$ of which $\varphi(z, \bar{z})$ is the derivative with respect to $\bar{z}$, and thus we have to prove the properties of $F(z, \bar{z})$ only in the specialized case

$$F(z, \bar{z}) = \frac{-1}{2\pi i} \int_D \frac{\partial \Phi}{\partial \bar{\zeta}} \frac{d\bar{\zeta}\, d\zeta}{\zeta - z}. \tag{64}$$

If we introduce the Newtonian kernel

$$V = \log |\zeta - z|^2,$$

then (64) is

$$F(z, \bar{z}) = \frac{-1}{2\pi i} \int_D \frac{\partial \Phi}{\partial \bar{\zeta}} \frac{\partial V}{\partial \zeta} d\bar{\zeta}\, d\zeta. \tag{65}$$

Now, formula (48) which we have claimed only for $k \geqq 2$ also holds for $k = 1$. Thus, for z in D, we have

$$\Phi(z, \bar{z}) = \frac{1}{2\pi i} \int_B \Phi \frac{d\zeta}{\zeta - z} + F(z, \bar{z}). \tag{66}$$

But the boundary integral is independent of $\bar{z}$, and this proves our lemma.

We are now going to prove Theorem 11. Since we are going to apply induction on k, we will have to prove our theorem in the sharper form that some solution of (61) depends analytically on any parameters occurring in f_β. We assume the existence of a solution of

$$\frac{\partial f}{\partial \bar{z}_\beta} = f_\beta(z_1, z_1; z_2, z_2, \cdots, z_k, z_k) \tag{67}$$

in $\beta = 2, \cdots, k$; with $z_1, \bar{z}_1$ as parameters. Denoting the solution of (67) by h and introducing the new function $g = f - h$, then the system (61) reduces to a system

$$\frac{\partial g}{\partial \bar{z}_1} = g_1(\bar{z}_1; z_1, z_2, \cdots, z_k)$$

$$\frac{\partial g}{\partial \bar{z}_2} = \cdots = \frac{\partial g}{\partial \bar{z}_k} = 0,$$

and this system is solvable by our lemma.

CHAPTER V

Meromorphic Functions in Euclidean Space

12. A simplified problem of Mittag-Leffler type. We are now returning to the topic of Chapter II, and the symbols $C_1, \cdots, C_r; B_1, \cdots, B_s; D_1, \cdots, D_t,$

D_∞ shall have the previous meaning. In addition to the incidence number ϵ_{pq} between C_p and B_q we also introduce the incidence number $\eta_{q\tau}$ between B_q and D_τ. An elementary combinatorial statement ("the boundary of a boundary is zero") implies the relation

$$\sum_{q=1}^{s} \epsilon_{pq}\eta_{q\tau} = 0 \tag{68}$$

and its validity is not impaired by the fact that the domains D_τ are not necessarily simplices. We furthermore introduce an arbitrary pointset S having the following properties: it is closed and nowhere dense, and if R is any domain then $R - S$ (that is $R - R \cdot S$) is also a domain. The set S will be termed an "exceptional" set. Obviously if a harmonic function φ in $R - S$ has an analytic continuation in all of R then the continuation is unique. If such a continuation exists we will say that "the singularities of φ at S are removable" or that "φ exists in (all of) R," or that "φ is analytic in R."

We now assume the existence of a harmonic function φ_τ in $\tilde{D}_\tau - S$, for each $\tau = 1, \cdots, t, \infty$, and we further assume that for each B_q the saltus

$$f_q = \sum_{\tau=1}^{\infty} \eta_{q\tau}\varphi_\tau \tag{69}$$

is analytic not only in $\tilde{B}_q - S$ but in all of $\tilde{B}_q$. We therefore may introduce the conglomerate function

$$F(x) = \frac{1}{\gamma_n} \sum_{q=1}^{s} \int_{B_q} \left(f_q \frac{\partial G}{\partial \xi_\alpha} - G \frac{\partial f_q}{\partial \xi_\alpha} \right) \sigma^\alpha . \tag{70}$$

Also, the saltus of $\{f_q\}$ in $\tilde{C}_p$ is defined as

$$f^p = \sum_q \epsilon_{pq} f_q = \sum_q \sum_\tau \epsilon_{pq}\eta_{q\tau}\varphi_\tau$$

and owing to relation (68) it turns out to be identically 0. Thus we may apply Theorems 1, 2, 3 and we obtain the following result.

THEOREM 12. *If the conglomerate function $\{\varphi_\tau\}$ is defined and harmonic in $\{\tilde{D}_\tau - S\}$, and if its saltus $\{f_p\}$ has none other than removable singularities in $\tilde{B}$, then the same saltus will be produced by the harmonic conglomerate function* (70) *which has no singularities whatsoever in $\{\tilde{D}_\tau\}$. Also, if $\{\varphi_\tau\}$ is a solution of a system of differential equations with constant coefficients,*

$$\Lambda\varphi = 0, \tag{71}$$

then $\{F_\tau\}$ is also a solution.

Thus, the functional elements

$$F_1 - \varphi_1 = F_2 - \varphi_2 = \cdots = F_\infty - \varphi_\infty$$

define a function Φ with the following properties. It is one functional element in $E_n - S$, it is harmonic and a solution of (71), *and in each $\tilde{D}_\tau$, the difference $\Phi - \varphi_\tau$ is analytic without exception.*

A seemingly striking situation arises, if the system (71) has the uniqueness property, and the set S is bounded. Being analytic outside a bounded domain, the function Φ must be analytic in the entire space and therefore $\varphi_\tau \equiv \Phi - F_\tau$ has only removable singularities in $\tilde{D}_\tau$ each. However, it should be noted that in this case, to start with, φ_∞ itself has to have an analytic continuation in the entire space. Now, let B'_λ be one of our faces lying in the boundary of D_∞. If it separates D_∞ from D_∞ then the saltus of φ is $\varphi_\infty - \varphi_\infty \equiv 0$, and thus naturally has no singularities. If however, B'_λ separates D_∞ from another domain D_c, then the saltus is $\pm(\varphi_\infty - \varphi_c)$ and if it is assumed to be free from singularities then so is φ_c itself. In other words: if we introduce the closet set $\bar{D}_1 + \cdots + \bar{D}_t$, and if we consider its interior D_0 and the frontier B_0; then our assumption requires *explicitly* that in a neighborhood $\tilde{B}_0$ the function $\{\varphi_\tau\}$ itself shall have no singularities, and only for "internal" faces B''_μ is the explicit requirement less stringent. If such a face separates two domains D_a, D_b then the assumption requires only that the saltus $\varphi_a - \varphi_b$ shall have no singularities. The theorem then allows the conclusion that actually the singularities of each φ_τ itself are removable everywhere in $\tilde{D}_\tau$.

13. A simplified problem of Weierstrass type. Everything else being as before, we consider a conglomerate function $\{\psi_\tau(x)\}$ with the following properties. Each $\psi_\tau(x)$ is analytic in $\tilde{D}_\tau$ and different from 0 in $\tilde{D}_\tau - S$. The "multiplicative" saltus

$$\prod_{\tau=1}^{\infty} \psi_\tau^{\eta_{q\tau}}$$

is by assumption analytic everywhere in $\tilde{B}_q$, and in the neighborhood of every point of $\tilde{D}_\tau - S$ the function $\log \psi_\tau(x)$ and its partial derivatives of first order both harmonic and a solution of (71), the latter having uniqueness property.

For fixed α, we introduce the conglomerate function

$$\varphi_\tau(x) = \frac{\partial}{\partial x_\alpha} \log \psi_\tau(x) = \frac{1}{\psi_\tau} \frac{\partial}{\partial x_\alpha} \psi_\tau,$$

and by Theorem 12 it is analytic in $\tilde{D}_\tau$. Thus, for fixed τ there exist analytic functions $F_1(x), \cdots, F_n(x)$ in $\tilde{D}_\tau$ such that

$$\frac{\partial \psi}{\partial x_\alpha} = \psi \cdot F_\alpha, \qquad \alpha = 1, \cdots, n.$$

On expanding ψ in a power series in a neighborhood of any point in $\tilde{D}_\tau$, the reader will easily convince himself that *every* $\psi_\tau(x)$ *is different from* 0 *in all of* $\tilde{D}_\tau$.

14. The theorem of Cousin. Returning to the topic of section 12, we now set up the modified integral

$$F^0(x) = \frac{1}{\gamma_n} \sum_{\mu=1}^{m} \int_{B''_\mu} \left(f_\mu \frac{\partial G}{\partial \xi_\alpha} - G \frac{\partial f_\mu}{\partial \xi_\alpha} \right) \sigma^\alpha$$

extended only over the "internal" faces B''_μ. Our present setup implies that we require the saltus function f_q to be analytic throughout $\tilde{B}_q$ only in case of internal faces, with no assumption made about the other faces.

Theorems 1 and 2 will permit the conclusion that for each D_a *whose closure is contained in* D_0 the corresponding functional element F^0_a has no singularities in $\tilde{D}_a$, and if two such simplices have a face B''_μ in common, then again

$$F^0_a - \varphi_a = F^0_b - \varphi_b . \tag{72}$$

Therefore if a number of such simplices $D_a , D_b , \cdots$ are the simplicial subdivision of a subdomain D^0 of D_0, then the functional elements (72) merge into a function Φ^0 in $\tilde{D}^0 - S$. However it must not be assumed that Φ^0 will be a solution of (71) because $\{\varphi_\tau\}$ is. All that can be said about the function (or functions)

$$h^0 = \Lambda\Phi^0 = \Lambda F^0 - \Lambda\varphi = \Lambda F^0$$

is that it is harmonic without singularities in D^0 (and indeed in $\tilde{D}^0$). We may now pose the problem of constructing a harmonic solution of the "unhomogeneous" system

$$\Lambda H = h^0$$

for any harmonic h^0 in any (or some specialized type of) domain D^0. If such a solution exists then $\Phi^0 - H$ will have in D^0 the same properties as were stated in Theorem 12 for the function itself.

A classical case arises if E_n is the space E_{2k} of the variables $z_\alpha = x_\alpha + iy_\alpha$, $\alpha = 1, \cdots , k$. Each φ_τ shall be a mereomorphic element in $\tilde{D}_\tau$, that is a quotient of two analytic functions in $z_1 , \cdots , z_k$ alone; with S being the union of the zero-sets of the denominators. The system (71) is

$$\frac{\partial\varphi}{\partial\bar{z}_1} = 0, \cdots , \frac{\partial\varphi}{\partial\bar{z}_k} = 0. \tag{73}$$

Theorem 11 will now give in substance a new proof for the classical theorem of Cousin about merging meromorphic elements in a polycylinder into one meromorphic function.

CHAPTER VI

Meromorphic Functions on Torus Space

15. Green's formula. We consider the locally Euclidean n-dimensional torus T_n with all periods 1:

$$T_n: \quad 0 \leqq x_\alpha < 1, \qquad \alpha = 1, \cdots , n.$$

Stokes' theorem (1) for ordinary or conglomerate functions remains in force. The scalar φ has to be strictly a function on the torus; that is, as a function in E_n it has to be strictly periodic in each variable x_α with period 1. However, any quantity ψ appearing only in a differential $d\psi$ is permitted to alter by an

additive constant if continued along a closed path in T_n. We now consider on the torus a function $G(\xi, x) = G(\xi - x)$ of the type (16). Formulas (14) and (15) are valid again, and if we apply them to the special domain $D = T_n$, $B = 0$, we obtain

$$\gamma_n f(x) = -\int_{T_n} \left(f \cdot \Delta G + \frac{\partial f}{\partial \xi_\alpha} \frac{\partial G}{\partial \xi_\alpha} \right) dV_\xi \tag{74}$$

$$\gamma_n f(x) = -\int_{T_n} (f \cdot \Delta G - G \cdot \Delta f)\, dV_\xi \tag{75}$$

On putting $f(\xi) = 1$ we obtain in particular

$$\gamma_n = -\int_{T_n} \Delta G \cdot dV_\xi ,$$

thus ΔG cannot be 0. But it might be a constant, and if so, the constant must of necessity be $-\gamma_n$. That is,

$$\Delta G = -\gamma_n . \tag{76}$$

Furthermore a solution of (76) must of necessity be unique. The actual existence of such a G can be proven in many ways. Suffice it to state here, that except for a constant factor, it is the sum of the series

$$\sum_{h_1, \cdots, h_n = -\infty}^{\infty\ \prime} \frac{\exp\{2\pi i[h_1(x_1 - \xi_1) + \cdots + h_n(x_n - \xi_n)]\}}{h_1^2 + \cdots + h_n^2} .$$

The series is not absolutely convergent but can be "summed" by general summation processes. From now on, G will be this solution of (76). Therefore, we now have

$$\begin{matrix} \gamma_n f(x) \\ 0 \end{matrix} = \int_B \left(f \frac{\partial G}{\partial \xi_\alpha} - G \frac{\partial f}{\partial \xi_\alpha} \right) \sigma^\alpha + \gamma_n \int_D f\, dV_\xi \tag{77}$$

and in particular

$$\begin{matrix} \gamma_n f(x) \\ 0 \end{matrix} = \int_B \left(f \frac{\partial G}{\partial \xi_\alpha} - G \frac{\partial f}{\partial \xi_\alpha} \right) \sigma^\alpha + \textit{constant}. \tag{78}$$

16. Green's integral. We again consider simplices $B_1, \cdots, B_s ; C_1, \cdots, C_r$. The complement $T_n - \bar{B}$ decomposes into domains $D_1, \cdots, D_t$, with no "D_∞" being present. We again set up the integral

$$F(x) = \frac{1}{\gamma_n} \int_B \left(f \frac{\partial G}{\partial \xi_\alpha} - G \frac{\partial f}{\partial \xi_\alpha} \right) \sigma^\alpha \tag{79}$$

for an arbitrary harmonic conglomerate function $f(\xi)$ in $\tilde{B}$, and we note that Theorems 1 and 2 carry over literally. In the proof to Theorem 1 relation (77)

has to be used instead of (20). In the proof to Theorem 2, the value of (25) is not 0, but some other constant, namely

$$-\gamma_n \sum_{q=0}^{\mu} \int_{B_q} f_q \, dV_\xi$$

and thus, in the proof, $\gamma_n(F(x) - F'(x))$ is constant, and hence $F(x)$ continuable.

The remarkable thing about the torus is the way in which Theorems 1 and 2 merge immediately with a part of Theorem 12.

THEOREM 13. *If $f(\xi)$ is a harmonic conglomerate function in $\tilde{B}$ and if its saltus vanishes identically in $\tilde{C}$, then f is in its turn the saltus of the harmonic conglomerate function F in $D \equiv T_n$.*

In particular, if S is an exceptional set, if φ is a conglomerate harmonic function in $\tilde{D} - S$, and if its saltus f has no singularities in $\tilde{B}$, then the function F is a harmonic conglomerate function without singularities having the same saltus as φ.

Finally we will analyze Theorem 3.

17. Meromorphic functions. Denoting again (79) by Lf we see on the basis of (31)–(34) that

$$L\frac{\partial f}{\partial \xi_\beta} - \frac{\partial}{\partial x_\beta} Lf = -\gamma_n \int_B f\sigma^\beta \tag{80}$$

and that in particular we have

$$L\frac{\partial f}{\partial \xi_\beta} - \frac{\partial}{\partial x_\beta} Lf = \text{constant}; \tag{81}$$

provided again that the saltuses of f, $\frac{\partial f}{\partial \xi_1}, \cdots, \frac{\partial f}{\partial \xi_n}$ vanish on C. Thus we will have

$$L\frac{\partial f}{\partial \xi_\beta} - \frac{\partial}{\partial x_\beta} Lf = 0$$

if and only if

$$\int_B f\sigma^\beta = 0. \tag{82}$$

It will now be easy for the reader to verify the following analogue to Theorem 3.

THEOREM 14. *If $f(\xi)$ is harmonic in $\tilde{B}$ and if its saltus vanishes identically in $\tilde{C}$, then*

$$L\Lambda f - \Lambda Lf = \text{constant} \tag{83}$$

If Λ be represented in some (or any) way as

$$\Lambda \equiv \sum_{\beta=1}^{n} \frac{\partial}{\partial x_\beta} \Lambda_\beta \tag{84}$$

where $\Lambda_1, \cdots, \Lambda_n$ are any operators, then

$$L\Lambda f - \Lambda L f = -\gamma_n \int_B \left(\sum_{\beta=1}^{n} \Lambda_\beta f \cdot \sigma^\beta \right). \tag{85}$$

In particular we have

$$L\Lambda f = \Lambda L f$$

if and only if

$$\int_B \left(\sum_{\beta=1}^{n} \Lambda_\beta f \cdot \sigma^\beta \right) = 0. \tag{86}$$

Therefore, if $\Lambda f = 0$ *and if* (86) *holds, then* ΛF *is likewise* 0.

For

$$\Lambda f = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2} \right) f, \qquad m < n$$

(86) is

$$\int_B \left(\sum_{\beta=1}^{n} \frac{\partial f}{\partial x_\beta} \sigma^\beta \right) = 0$$

Or, if

$$\Lambda f = \frac{\partial f}{\partial x_1} + i \frac{\partial f}{\partial x_2} = 2 \frac{\partial f}{\partial \bar{z}},$$

where

$$z = x_1 + i x_2 ,$$

then we may put $\Lambda_1 = 1, \Lambda_2 = i, \Lambda_3 = \cdots = \Lambda_n = 0$, and therefore

$$\sum_{\beta=1}^{n} \Lambda_\beta f \cdot \sigma^\beta = f \cdot (d\xi_1 - i d\xi_2) \cdot \sigma^{1,2} \equiv f \, d\bar{\xi} \cdot \sigma^{1,2}.$$

Turning now to the torus space T_{2k} of the complex variables $z_\alpha = x_\alpha + i y_\alpha$, $\alpha = 1, \cdots, k$, we therefore obtain the following analogue to Theorem 12.

THEOREM 15. *If* $\varphi_\tau(z_1, \cdots, z_k)$ *is meromorphic in* $\tilde{D}_\tau$, $\tau = 1, \cdots, l$, *but each* $\varphi_a - \varphi_b$ *is analytic in* $\tilde{B}_q$ *and if the k periodicity conditions*

$$\int_B f \, d\bar{\xi}_\alpha \cdot \omega^{k-1} = 0, \qquad \alpha = 1, \cdots, k \tag{87}$$

are fulfilled, then there exists a meromorphic function $F(z_1, \cdots, z_k)$ *on the torus, such that* $F - \varphi_\tau$ *is analytic in* $\tilde{D}_\tau$.

If the conditions (87) are not stipulated, then by (83) we have at any rate

$$\frac{\partial F}{\partial \bar{z}_\alpha} = c_\alpha , \qquad \alpha = 1, \cdots, k$$

where c_α is a constant. In this case the function

$$F(z_1, \cdots, z_k ; \bar{z}_1, \cdots, \bar{z}_k) - c_1\bar{z}_1 - \cdots - c_k\bar{z}_k$$

in the plane will be strictly meromorphic, however if continued along a closed path on the torus it alters by an additive constant which need not be 0.

The additive constants do not alter the algebraic singularities, and in this sense we have

THEOREM 16. *If an exceptional set S on T_{2k} is locally the singular set of a meromorphic function, then it has the same property in the large.*

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DISCONTINUOUS GROUPS

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I. INTRODUCTION

1. Let G be a topological group, and let H be a *discrete* subgroup of G; this means that there exists in G a neighborhood U of the unit element e such that no other element of H is contained in U. If M is a subset of G and $a \in H$, then the set Ma is an *image* of M. A *fundamental set* F relative to H is defined by the following three properties: 1) $FH = G$; 2) $Fa \cap F = 0$, whenever $e \neq a \in H$; 3) F is a Borel set in G. The first two properties establish that every point x of G is covered by one and only one image of F. We obtain an arbitrary set M with these two properties, if we select a representative x from every left coset xH of H relative to G; however, in general, such a set M is not a Borel set. In Lemma 2, we shall prove that a fundamental set exists, if G satisfies the second axiom of countability.

We say that a fundamental set F is *normal*, if every point of G has a neighborhood contained in the union of a finite number of images of F. An image Fa of F is called a *neighbor* of F, if $\bar{F}a \cap \bar{F} \neq 0$. Let H_0 be the group generated by all elements $a \in H$ satisfying $\bar{F}a \cap \bar{F} \neq 0$. In section 9 we shall prove, for any connected group G satisfying the second axiom of countability, that $H_0 = H$, whenever the fundamental set F is normal; in particular, if F has only a finite number of neighbors, then H possesses a finite system of generators.

2. Let T be a topological space of points τ, and let $\tau \to f(\tau, x)$, $x \in G$, be an open continuous *representation* of G as a transitive group of homeomorphic mappings of T on itself. The representation $\tau \to f(\tau, a)$, $a \in H$, of H in T is called *discontinuous*, if no sequence $f(\tau, a_n)$ $(n = 1, 2, \cdots)$ converges to a point in T, as a_n runs over distinct elements of H. Let τ_1 be a given point of T, and consider the set C of elements $x \in G$ satisfying $f(\tau_1, x) = \tau_1$; obviously C is a closed subgroup of G. It is known that then T and $C\backslash G$ are homeomorphic, where the topological space $C\backslash G$ consists of the right cosets Cy of C relative to G; on the other hand, for any closed subgroup C of G and any homeomorphic mapping of $C\backslash G$ on a topological space T, the transformations $Cy \to Cyx$, $x \in G$, define an open continuous representation of G as a transitive group of homeomorphic mappings of T on itself. Therefore, in order to find all discontinuous representations of H, we can restrict ourselves to the investigation of the case $T = C\backslash G$, where C is any closed subgroup of G. In Lemma 6 we shall prove that the representation of H in $C\backslash G$ is discontinuous in the case of a compact group C.

Let G be a locally compact group satisfying the second axiom of countability. For all Borel sets B in G, the Haar measure defines a completely additive and right-invariant volume $v(B)$, which is positive for all open sets. It will be proved that the volume of a fundamental set F does not depend upon the parti-

cular choice of F and only upon the discrete subgroup H. The main result of the second chapter is the following theorem.

THEOREM 1: *Let $v(F)$ be finite; then the representation of H in $C\backslash G$ is discontinuous, if and only if the closed subgroup C is compact.*

The interest of Theorem 1 lies in the *necessity* of the condition concerning C. The proof uses an idea similar to that in the proof of Poincaré's theorem in ergodic theory.

3. Let the subgroup C be compact, and assume that the representation of H in $C\backslash G$ has no fixed points; this means that $Cya \neq Cy$ for all $y \in G$ and all $a \neq e$ in H. It can readily be seen that this assumption is fulfilled, in particular, if H does not contain any element of finite order except e. We shall prove, in Lemma 7, that there exists a fundamental set F relative to H such that $F = CF$, whenever G satisfies the second axiom of countability; then F may be considered as a fundamental set in the space $C\backslash G$.

Let G be a locally compact group satisfying the second axiom of countability. We say that H is a subgroup of the *first kind* in G, if there exists a normal fundamental set relative to H, of finite volume, having only a finite number of neighbors. We have no general constructive method of deciding whether a given subgroup H is of the first kind; however, in the known particular cases where we are able to answer this question, the problem is simplified by considering it for the space $C\backslash G$ instead of G, with suitably chosen compact C. In the case of the *unimodular* group, e.g., $G = G_m$ is the multiplicative group of all real m-rowed matrices x with determinant ± 1, and $H = H_m$ is the subgroup consisting of all $x = u$ with integral elements; if C denotes the orthogonal group in m dimensions, then the space $C\backslash G$ is homeomorphic to the space T of the positive symmetric matrices $t = x'x$ with determinant 1, and H_m is represented by the transformations $t \to u'tu$ of T into itself. In this way, Minkowski replaced the problem of constructing a fundamental set for H_m in G_m by the corresponding problem concerning the reduction of positive quadratic forms in m variables, and he proved that H_m is a subgroup of the first kind in G_m.

The problem of finding *all* subgroups H of the first kind in G seems to be far beyond our power, even if we consider only the particular case $G = G_m (m \geqq 2)$. The solution is known only for $m = 2$:

G_2 is the group of all two-rowed matrices $\begin{pmatrix} ab \\ cd \end{pmatrix}$ with real elements and the determinant 1. A subgroup H of G_2 is of the first kind, if and only if the representation of H by the linear mappings $z \to (az + b)(cz + d)^{-1}$ possesses in the upper z half-plane a fundamental polygon with a *finite* number of vertices. However, even in this comparatively simple case, we have no method of deciding whether two arbitrarily given subgroups H and H_0 of G_2 are isomorphic.

4. In the third chapter, we investigate the properties of a special type of discrete groups; these groups include the unimodular group H_m, and they are found by the following considerations:

We assume that the topological group G may be imbedded in a connected topological ring A which is locally compact and satisfies the second axiom of countability. It follows from a theorem of Jacobson and Taussky[1] that A is isomorphic to an algebra of finite rank in R, the field of real numbers. Consequently, we can introduce the norm $N(x)$ of an arbitrary element x in A. Let G be the group of all x with $N(x) = \pm 1$.

On the other hand, let A_0 be a simple algebra of finite rank in R_0, the field of rational numbers. Under extension of R_0 into R, the simple algebra A_0 in R_0 is replaced by a semi-simple algebra in R, and we assume now that this is the algebra A. Let an *order* J in A_0 be given; a quantity a is called a *unit* in J, if both a and a^{-1} belong to J. It is obvious that the units in J constitute a discrete subgroup H in G.

THEOREM 2: *The group of units in a simple order is of the first kind.*

The proof of this theorem depends upon the theory of reduction of positive quadratic forms; it is a generalization of Minkowski's proof concerning the unimodular group H_m.

5. By a well-known theorem of Wedderburn, the simple algebra A_0 is isomorphic to an algebra of matrices whose elements are arbitrary quantities in a division algebra D_0. Let Z be the center of D_0. In two important special cases the theory of the group of units in J had been investigated a long time since, namely in the case $A_0 = Z$ by Dirichlet, and in the case $D_0 = R_0$ by Minkowski. Recently, Humbert[2] studied the more general case $D_0 = Z$, and Weyl[3] proved Humbert's results as simple consequences of a geometric theory of reduction of lattices, which he applied also in the case of a totally definite quaternion algebra D_0 over a totally real center Z.

Concerning the general case of the group H of units in an arbitrary simple order, Eichler[4] stated that H has a finite system of generators; however, his proof is correct only for $A_0 = D_0$. Moreover, Eichler's paper contains an interesting theorem about continuous mappings of G/H on manifolds with the Poincaré group H, also without complete proof; it is related to Theorem 1, and it was the starting-point of the researches in the second chapter.

6. Let an *involution* in A_0 be given, i.e., a mapping $x \to x^*$ of A_0 onto itself such that $(x^*)^* = x$, $(x + y)^* = x^* + y^*$, $(xy)^* = y^* x^*$; under extension of R_0 into R, we obtain an involution in A. Let $s \in A_0$, $N(s) \neq 0$ and $s^* = s$, and consider the set $G(s)$ of all $x \in A$ such that $x^* s x = s$; obviously $G(s)$ is a subgroup

[1] N. Jacobson and O. Taussky, *Locally compact rings*, Proc. Nat. Acad. Sci. 21, pp. 106–108 (1935).

[2] P. Humbert, *Théorie de la réduction des formes quadratiques définies positives dans un corps algébrique K fini*, Comment. Math. Helv. 12, pp. 263–306 (1940).

[3] H. Weyl, *Theory of reduction for arithmetical equivalence. II*, Trans. Amer. Math. Soc. 51, pp. 203–231 (1942).

[4] M. Eichler, *Zur Einheitentheorie der einfachen Algebren*, Comment. Math. Helv. 11, pp. 253–272 (1939).

of G. For any given order J in A_0, the intersection $H \cap G(s)$ defines a discrete subgroup $H(s)$ of $G(s)$.

Assume now that the involution $x \to x^*$ leaves invariant all elements of the center Z. By the results of Albert,[5] the division algebra D_0 is then either Z itself or a quaternion algebra over Z.

In the first case, A_0 is the ring of all m-rowed matrices x with elements in Z, and x^* is the transpose of x; then $H(s)$ is the group of units of the symmetric matrix s. In particular, let $Z = R_0$; then it is known[6] that $H(s)$ is a subgroup of the first kind in $G(s)$, except in the trivial case $m = 2$, $N(s) = -r^2$, r a rational number. This result was proved by an application of Hermite's method of continuous reduction of indefinite quadratic forms: We consider the space T of all positive symmetric matrices t, with real elements, satisfying $s = ts^{-1}t$; the transformations $t \to x^*tx$, $x \in G(s)$, are transitive in T, and the group C of all x satisfying $t = x^*tx$ is compact, for any fixed t; from the theory of reduction of positive quadratic forms it follows that the representation $t \to a^*ta$, $a \in H(s)$, is of the first kind in T, except in the above mentioned trivial case. If n, $m - n$ is the signature of s, then T has $n(m - n)$ dimensions; it can easily be shown, by Theorem 1, that any discontinuous representation of $H(s)$ is at least $n(m - n)$-dimensional.

In the second case, A_0 is the ring of all m-rowed matrices x with elements in the quaternion algebra D_0 over Z, and x^* is the conjugate transpose of x. In particular, let Z be totally real, of degree h over R_0, and let the norm, relative to Z, of every element $\neq 0$ in D_0 be positive at $h - 1$ infinite prime spots of Z. It is known[7] that then $H(e)$ is of the first kind in $G(e)$; it can easily be proved, by Theorem 1, that any discontinuous representation of $H(e)$ is at least $m(m + 1)$-dimensional.

These examples indicate that a systematic investigation of all unit groups of fixed points in an involution might be of some interest.

II. General Theory

7. Throughout the present chapter, G is a topological group and H is a discrete subgroup of G.

Lemma 1: *For every $x \in G$ there exists a neighborhood V_x of x such that $V_x a \cap V_x = 0$, for all elements $a \neq e$ in H.*

Proof: Let U be a neighborhood of e containing no other element a of H. Since $e^{-1}e = e$, there exists a neighborhood V of x such that $V^{-1}V \subset U$. Define $V_x = xV$, and let y, z be two arbitrary points in V_x; then $y^{-1}z \in U$, hence $y^{-1}z \neq a$, $ya \neq z$, and V_x has the required property.

Lemma 1 establishes that the neighborhood V_x of x contains at most one element from every left coset yH of H relative to G.

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[5] A. A. Albert, *Structure of algebras*, New York (1939); Chap. X.

[6] C. L. Siegel, *Einheiten quadratischer Formen*, Abh. Math. Sem. Hansischen Univ. 13, pp. 209–239 (1940).

[7] C. L. Siegel, *Symplectic geometry*, Amer. Jour. Math. 55, pp. 1–86 (1943).

LEMMA 2: *Let G satisfy the second axiom of countability; then there exists a fundamental set relative to H.*

PROOF: Let $A_1, A_2, \cdots$ constitute a basis of G. We consider all A_k contained in at least one of the neighborhoods V_x defined in Lemma 1, and we denote these A_k by $B_1, B_2, \cdots$. Let

$$F_1 = B_1, \qquad F_k = B_k - (B_1 \cup \cdots \cup B_{k-1})H \qquad (k = 2, 3, \cdots),$$
$$F = F_1 \cup F_2 \cup \cdots.$$

Since any V_x is the union of a certain number of sets B_k, every element x of G is contained in some B_k; in particular, this is true for every element a of H. On the other hand, by Lemma 1, B_k does not contain two elements of H. Consequently, H is finite or countably infinite, and F is a Borel set.

We have $F_k \subset B_k (k = 1, 2, \cdots)$; therefore, by Lemma 1,

$$F_k a \cap F_k = 0 \qquad (e \neq a \in H);$$

moreover

$$F_k a \subset (B_1 \cup \cdots \cup B_{l-1})H, \qquad F_k a \cap F_l = 0 \qquad (1 \leqq k < l;\ a \in H);$$

hence $F_k a \cap F_l = 0$, for arbitrary positive integers k, l and $e \neq a \in H$. I[t] follows that $Fa \cap F = 0$, for $e \neq a \in H$.

Let x be any element of G. Since x lies in some B_k, there exists also an index k such that x lies in $B_k H$, but not in $(B_1 \cup \cdots \cup B_{k-1})H$; then $x \in B_k H - (B_1 \cup \cdots \cup B_{k-1})H = F_k H$, $x \in FH$. It follows that $FH = G$.

We have verified that F has the three characteristic properties of a fundamental set relative to H, and the lemma is proved.

If $S_1, S_2, \cdots$ is a finite or countably infinite number of disjoint sets, we denote their union by the sign $\sum$.

LEMMA 3: *Let G satisfy the second axiom of countability, and let E, F be two fundamental sets relative to H; then there exist two decompositions*

$$E = \sum_{a \in H} E_a, \qquad F = \sum_{a \in H} F_a,$$

where E_a and $F_a = E_a a$ are Borel sets.

PROOF: Define $Ea \cap F = F_a$, $E \cap Fa^{-1} = E_a$; then E_a and $F_a = E_a a$ are Borel sets; furthermore

$$F_a \cap F_b = Ea \cap Eb \cap F = 0 \qquad (a \in H; b \in H; a \neq b),$$
$$E_a \cap E_b = E \cap Fa^{-1} \cap Fb^{-1} = 0$$

and

$$\sum_{a \in H} F_a = \Big(\sum_{a \in H} Ea\Big) \cap F = (EH) \cap F = G \cap F = F,$$
$$\sum_{a \in H} E_a = E \cap \Big(\sum_{a \in H} Fa^{-1}\Big) = E \cap (FH) = E \cap G = E;$$

q.e.d.

8. Let us assume that there exists a fundamental set F, relative to H, such that $\bar{F}$ is compact, and consider any infinite set of cosets xH, $x \in S$. We choose $a_x \in H$ such that $xa_x \in F$; then the set xa_x, $x \in S$, has a limit point y in $\bar{F}$, hence the set of points xH, $x \in S$, in G/H has the limit point yH; i.e., G/H is compact. Conversely, we shall prove that the existence of a fundamental set with compact closure follows again from the compactness of G/H, provided G satisfies the second axiom of countability. However, in the following lemma, we do not need the latter assumption.

LEMMA 4: *Let the topological space G/H be compact; then G is locally compact.*

PROOF: Let U be a neighborhood of e in G containing no other element of H. Since the topological space G is regular, we can choose a neighborhood W of e such that $\overline{W}^{-1}W\overline{W}^{-1}\overline{W} \subset U$. Let S be an infinite subset of $\overline{W}$. Since G/H is compact, there exists a point z in G such that, for any neighborhood V of z in G, the set VH contains infinitely many points of S. Let $V \subset Wz$, and let x, y be two points of $VH \cap S$; then xa and yb are points of Wz, for suitably selected elements a and b of H, and

$$ab^{-1} \in x^{-1}W\overline{W}^{-1}y \subset \overline{W}^{-1}W\overline{W}^{-1}\overline{W} \subset U;$$

hence $a = b$. It follows that there exists a fixed $a \in H$ such that Va contains infinitely many points of S, for any neighborhood V of z; then S has the limit point za, and $\overline{W}$ is compact. Hence G is locally compact; q.e.d.

Assume now that G satisfies the second axiom of countability, and consider again the construction of F in the proof of Lemma 2. In view of Lemma 4, we can suppose that the sets $\bar{V}_x$ are compact; then also the sets $\bar{B}_k$ and $\bar{F}_k$ are compact. We shall prove that F_k is empty for all sufficiently large values of k. Otherwise there would exist a sequence of points $x_1, x_2, \cdots$ in G such that $x_n \in F_{k_n}$, where $k_1, k_2, \cdots$ is an increasing sequence of indices, and a sequence of points $a_1, a_2, \cdots$ in H such that the sequence $x_n a_n (n = 1, 2, \cdots)$ converges to a point x_0 in G. The point x_0 lies in some B_k. If $x_0 \in B_l$ and $k > l$, then

$$B_l \cap F_k H = B_l \cap (B_k H - (B_1 \cup \cdots \cup B_{k-1})H) = 0;$$

consequently, no $x_n a_n (k_n > l)$ lies in B_l, and this is a contradiction. It follows that $F = F_1 \cup F_2 \cup \cdots$ is the union of a finite number of F_k; hence $\bar{F}$ is also compact.

Let $\bar{F}$ be compact, and let C be an arbitrary compact subset in G. Consider all $a \in H$ such that $\bar{F}a \cap C \neq 0$. All these a are contained in the compact set $\bar{F}^{-1}C$; since H is discrete, they belong to a finite set. On the other hand, the images Fa cover the whole space G. It follows that C is completely covered by a finite number of images Fa. Since G is locally compact, every point in G has a neighborhood contained in a finite number of images of F. Furthermore, for $C = \bar{F}$, we see that F has only a finite number of neighbors Fa.

9. In this section we assume that there exists a normal fundamental set F; this means that a suitably chosen neighborhood of every point is contained

in the union of a finite number of images of F. It is obvious that then every compact set in G is covered by a finite number of images of F. Let A be the set of all elements a of H which are contained in $\bar{F}^{-1}\bar{F}$; it is clear that $a \in A$, if and only if Fa is a neighbor of F. Let H_0 be the smallest group containing A; i.e., H_0 is the group generated by all $a \in A$. We are going to prove that $H_0 = H$, provided G is connected.

If $H_0 \neq H$, let $G_0 = \sum_{a \in H_0} Fa$; then $G_0 \neq 0$ and $G - G_0 = \sum_{b \in H - H_0} Fb \neq 0$. Since G is connected, the two sets $\bar{G}_0$ and $\overline{G - G_0}$ have a common point x. On the other hand, only a finite number of images of F enter into a suitably chosen neighborhood of x. It follows that there exist a point a in H_0 and a point b in $H - H_0$ such that $x \in \bar{F}a$ and $x \in \bar{F}b$; then $ba^{-1} \in \bar{F}^{-1}\bar{F}$, $ba^{-1} \in A \subset H_0$, $b \in H_0$, and this is a contradiction.

10. In this section we assume that G is locally compact and satisfies the second axiom of countability. The volume $v(B)$ of a Borel set B in G has the following properties:[8] $v(\sum_k B_k) = \sum_k v(B_k)$; $v(Bx) = v(B)$, for all $x \in G$; $v(xB) = \Delta(x)v(B)$, where $\Delta(x)$ is a positive continuous function of x and $\Delta(xy) = \Delta(x)\Delta(y)$, for all $x, y \in G$; $v(B) > 0$, for all open sets B; $v(B)$ is finite for all compact sets B. Moreover, by these properties, $v(B)$ is uniquely determined up to a positive constant factor.

Let E, F be two fundamental sets relative to H. Any Borel set in G/H can be expressed in the form BH, where B is a Borel set in G; let $BH \cap E = C$, $BH \cap F = D$. In view of Lemma 3,

$$C = \sum_{a \in H} (BH \cap E_a), \qquad D = \sum_{a \in H} (BH \cap F_a), \qquad F_a = E_a a$$

$$v(C) = \sum_{a \in H} v(BH \cap E_a), \qquad v(D) = \sum_{a \in H} v(BH \cap F_a), \qquad v(BH \cap E_a) = v(BH \cap F_a);$$

hence $v(C) = v(D)$, and we can define, for all Borel sets BH in G/H, a volume in the space G/H by the formula $v_H(B) = v(BH \cap F)$. Then $v_H(B) \leqq v(F)$, and $v_H(G) = v(F) = v(E)$.

LEMMA 5: *Let $v(F)$ be finite; then $v(B)$ and $v_H(B)$ are left-invariant.*

PROOF: It follows from the definition of a fundamental set F that also $E = xF$ is a fundamental set, for any given x in G. Hence

$$v(F) = v(E) = \Delta(x)v(F).$$

We infer from the construction of the particular fundamental set F in the proof of Lemma 2 that it contains the open set B_1; hence $v(F) > 0$. On the other hand, we assumed $v(F)$ to be finite. Therefore $\Delta(x) = 1$, for all $x \in G$, and

$$v_H(xB) = v(xBH \cap E) = \Delta(x)v(BH \cap F) = v_H(B).$$

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[8] J. von Neumann, *The uniqueness of Haar's measure*, Rec. Math. N. S. (Mat. Sbornik) 1 (43), pp. 721–734 (1936). [English. Russian summary.]

11. LEMMA 6: *Let C be a compact subgroup of G; then the representation of H in $C\backslash G$ is discontinuous.*

PROOF: Let $a_1, a_2, \cdots$ be a sequence of distinct elements in H, let $x \in G$, and assume that the set of right cosets Cxa_n $(n = 1, 2, \cdots)$ has the limit point Cy in $C\backslash G$. We choose a neighborhood P of y and a neighborhood Q of e such that $P^{-1}Q^{-1}QP$ contains no element $\neq e$ of H. Then $xa_n = c_n p_n$, $c_n \in C$, and $p_n \in P$ for all sufficiently large n. Since C is compact, there exists a point $c \in C$ such that $c_n \in cQ$ for infinitely many n. Let $c_n \in cQ$, $c_m \in cQ$, $m \neq n$, then $a_m^{-1}a_n = p_m^{-1}c_m^{-1}c_n p_n \in P^{-1}Q^{-1}QP$; hence $a_m = a_n$, and this is a contradiction. It follows that the representation of H in $C\backslash G$ is discontinuous.

PROOF OF THEOREM 1: In view of Lemma 6 we have only to prove the necessity of the compactness of C. We assume that the representation of H in $C\backslash G$ is discontinuous, where C is a given closed subgroup of G. Consider the particular fundamental set F constructed in the proof of Lemma 2; it contains the open set B_1, hence it contains also an open set B_0, whose closure $\bar{B}_0 = B$ is compact. Then $v_H(B) = v(BH \cap F) \geqq v(B_0H \cap F) = v(B_0) > 0$. Consequently there exists a compact set B such that the volume $v_H(B)$ is positive.

Let $c_n (n = 1, 2, \cdots)$ be a sequence of distinct points in C, and define

$$c_n^{-1}BH \cap F = P_n, \qquad P_n \cup P_{n+1} \cup \cdots = Q_n \ (n = 1, 2, \cdots),$$

$$Q_1 \cap Q_2 \cap \cdots = Q.$$

Then P_n, Q_n, Q are Borel sets, $Q_n \subset F$, and by Lemma 5

$$v(F) \geqq v(Q_n) \geqq v(P_n) = v_H(c_n^{-1}B) = v_H(B);$$

moreover

$$\sum_{k=1}^{n-1} v(Q_k - Q_{k+1}) = v(Q_1) - v(Q_n) \leqq v(F) \qquad (n = 1, 2, \cdots)$$

$$\sum_{k=1}^{\infty} v(Q_k - Q_{k+1}) = v(Q_1) - v(Q);$$

hence

$$v(Q) = \lim_{n \to \infty} v(Q_n) \geqq v_H(B) > 0.$$

It follows that Q is non-empty.

Let $x \in Q$; then $x \in P_n$ for an infinite number of indices n, hence $x \in c_n^{-1}BH$ for the same set of indices. Consequently, there exists a sequence $a_n \in H (n = 1, 2, \cdots)$ such that $c_n x a_n \in B$ for infinitely many n. But B is compact and the representation of H in $C\backslash G$ is discontinuous. Therefore it follows from $Cxa_n \in CB$ that a_n belongs to a finite set of elements in H. This implies that $c_n x a \in B$, for a fixed $a \in H$ and infinitely many n, and hence the set $c_n (n = 1, 2, \cdots)$ has a limit point c_0. Since C is closed, c_0 is a point of C. This proves the compactness of C.

12. Let C be a compact subgroup of G. The representation of H in $C\backslash G$ has no fixed point, if and only if none of the conjugate subgroups $x^{-1}Cx$, $x \in G$, contains an element $a \neq e$ of H. This assumption is satisfied, in particular, if H does not contain any element of finite order except e: Let $a \in x^{-1}Cx$, $a \in H$, then $a^n \in x^{-1}Cx$ $(n = 1, 2, \cdots)$; since $x^{-1}Cx$ is compact and H is discrete, it follows that a is an element of finite order, hence $a = e$.

If a fundamental set F has the property $F = CF$, we may consider it also as a set in $C\backslash G$.

LEMMA 7: *Let G satisfy the second axiom of countability; then a fundamental set in $C\backslash G$ relative to H exists, if and only if the representation of H in $C\backslash G$ has no fixed point.*

PROOF: The necessity of the condition is easily proved: Let $F = CF$ be a fundamental set in $C\backslash G$ relative to H, and let $a \in x^{-1}Cx$, $a \in H$. Choose $b \in H$ such that $xb \in F$; then $b^{-1}ab \in (xb)^{-1}C(xb) \subset F^{-1}C\,F = F^{-1}F$, $F(b^{-1}ab) \cap F \neq 0$; hence $b^{-1}ab = e$, $a = e$. In this part of the proof, we did not use the assumption that G satisfies the second axiom of countability.

Conversely, assume that the representation of H in $C\backslash G$ has no fixed point. Let $x \in G$ and $y \in C$. Since $x^{-1}yx \neq a$, for all elements $a \neq e$ of H, there exist a neighborhood $P_{x,y}$ of x and a neighborhood $Q_{x,y}$ of y such that $P_{x,y}^{-1}Q_{x,y}P_{x,y}$ does not contain any element $\neq e$ of H. For any given x, the compact set C can be covered by a finite number of the neighborhoods $Q_{x,y}$, and the corresponding $P_{x,y}$ have a non-empty open intersection V_x. Consequently every point x in G has a neighborhood V_x such that $CV_xa \cap CV_x = 0$, for all $a \neq e$ in H.

Now we generalize the proof of Lemma 2 in the following way. We define

$$F_1 = CB_1, \qquad F_k = CB_k - (CB_1 \cup \cdots \cup CB_{k-1})H \qquad (k = 2, 3, \cdots),$$

$$F = F_1 \cup F_2 \cup \cdots.$$

Since $C(CA \cup CB) = CA \cup CB$ and $C(CA - CB) = CA - CB$, for arbitrary sets A and B, it is readily proved that F is a fundamental set and that $F = CF$.

13. In this section we assume that G is a locally compact group satisfying the second axiom of countability. Let C be a compact subgroup of G. Any Borel set in $C\backslash G$ can be expressed in the form CB, where B is a Borel set in G. The formula $v_c(B) = v(CB)$ defines in $C\backslash G$ a right-invariant completely additive volume, which is positive for open sets and finite for compact sets; it is uniquely determined by these properties, up to an arbitrary positive constant factor.[9] In order to establish that H is a subgroup of the first kind in G, it is sufficient to prove the existence of a normal fundamental set $F = CF$ in $C\backslash G$, of finite volume $v_c(F)$, having only a finite number of neighbors.

[9] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris (1940); pp. 42–45.

By the result of section 8, the group H is certainly of the first kind, whenever G/H is compact.

In the next chapter we shall apply the following lemma:

LEMMA 8: *The group H is of the first kind, if and only if there exists, for some compact subgroup C of G, an open set $M = CM$ in $C\backslash G$ of finite volume $v_c(M)$ such that $MH = G$ and that $Ma \cap M \neq 0$ holds only for a finite number of $a \in H$.*

PROOF: Let H be of the first kind; this means that there exists a normal fundamental set F of finite volume $v(F)$, having only a finite number of neighbors. Let Q be the union of all open sets contained in $\sum_{a \in A} \bar{F}a$, where A denotes the finite set of all elements of H contained in $\bar{F}^{-1}\bar{F}$; the set Q is open. If x is an arbitrary point of F, then a certain neighborhood W of x is covered by the union of a finite number of images Fa, $a \in H$; on the other hand, if Fa enters into every neighborhood of x then Fa is a neighbor of F and $a \in A$; consequently there exists also a neighborhood $W \subset Q$. It follows that $x \in Q$, $F \subset Q$, $QH = G$. Furthermore,

$$Qa \cap Q \subset \bigcup_{b,c \in A} (\bar{F}ba \cap \bar{F}c);$$

hence $Qa \cap Q = 0$ for any element $a \in H$ not lying in the finite set $A^{-1}AA$. Therefore $M = Q$ and $C = e$ have the required properties.

Conversely, let C be a compact subgroup of G, and let $M = CM$ be open, $v_c(M)$ finite, $MH = G$, $Ma \cap M \neq 0$ for only a finite set S of $a \in H$. In view of $MH = G$, we can determine, by the construction in the proof of Lemma 2, the fundamental set F as a subset of M; then $v(F) \leqq v(M) = v_c(M)$, hence $v(F)$ is finite. Let x be an arbitrary point in G, and let $x \in Mb$, $b \in H$; then b belongs to a finite set S_x. The intersection of all Mb, $b \in S_x$, is a neighborhood W_x of x. If $Fa \cap W_x \neq 0$, $a \in H$, then $Ma \cap W_x \neq 0$, $Ma \cap Mb \neq 0$, for $b \in S_x$; hence a belongs to the finite set SS_x. It follows that F is normal. Moreover, let $c \in H$, and let $\bar{M}c$, $\bar{M}$ have a common point x. We choose $a \in H$ such that $x \in Ma$; then $Ma \cap Mc \neq 0$ and $Ma \cap M \neq 0$, hence $ac^{-1} \in S$ and $a \in S$; this proves that c belongs to the finite set $S^{-1}S$. Since $\bar{F} \subset \bar{M}$, the fundamental set F has only a finite number of neighbors. Consequently the group H is of the first kind in G.

III. GROUPS OF UNITS IN SIMPLE ORDERS

14. Let A_0 be a simple algebra of finite rank in R_0, the field of rational numbers; then A_0 is isomorphic to an algebra of n-rowed matrices $\xi = (\xi_{kl})$, whose elements ξ_{kl} $(k, l = 1, \cdots, n)$ are arbitrary quantities in a division algebra D_0. The center Z of D_0 is an algebraic number field; let h denote the degree of Z over R_0, and let g be the rank of D_0 in R_0; then $g/h = s^2$ is the square of a positive rational integer s. We choose a basis $\delta_1, \cdots, \delta_g$ of D_0 in R_0 and extend R_0 into R, the field of real numbers; then D_0 is extended into a semi-simple algebra D in R. Every element of D is uniquely expressed by the linear form $\delta = \delta_1 x_1 + \cdots + \delta_g x_g$ with arbitrary real $x_1, \cdots, x_g$; we say that $x_1, \cdots, x_g$ are

the coordinates of δ. In the regular representation of D, the element δ is represented by a g-rowed matrix $\hat{\delta}$. The g-rowed unit matrix will be denoted by ϵ.

Let r_2 be the number of imaginary infinite prime spots of Z, and let D_0 be ramified at r_3 real infinite prime spots of Z; define $r_1 = h - 2r_2 - r_3$. There exists a non-singular g-rowed matrix c with constant complex elements such that $c\hat{\delta}c^{-1}$ decomposes into s equal matrices λ of degree hs, and λ itself breaks up into h matrices $\lambda_1, \cdots, \lambda_h$ of degree s; the g elements of these matrices are homogeneous linear functions of $x_1, \cdots, x_g$, whose matrix is the inverse of c'; the r_1 matrices $\lambda_1, \cdots, \lambda_{r_1}$ are real; the r_2 pairs of matrices λ_{r_1+k} and $\lambda_{r_1+r_2+k}$ ($k = 1, \cdots, r_2$) are conjugate complex; the r_3 matrices λ_k ($k = r_1 + 2r_2 + 1, \cdots, h$) have the form $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, where α and β denote complex matrices of degree $s/2$; moreover, the matrix $c'\bar{c}$ is real. In order to simplify the notation we write $c\hat{\delta}c^{-1} = \delta$.

The general quantity of the algebra A_0 in R, or A, is the n-rowed matrix $\xi = (\xi_{kl})$ with arbitrary elements ξ_{kl} from D; we define $(\hat{\xi}_{kl}) = \hat{\xi}$ and consider ξ as the n-rowed matrix with the g-rowed matrix elements $c\hat{\xi}_{kl}c^{-1} = \xi_{kl}$ ($k, l = 1, \cdots, n$). The gn^2 coordinates of the ξ_{kl} form the coordinates of ξ. Let G be the set of all ξ lying on the surface $|\xi| = \pm 1$. If we introduce the natural topology of R for all coordinates of ξ, the set G becomes a locally compact topological group satisfying the second axiom of countability. Let C be a *maximal* compact subgroup of G. Since the elements of C are matrices, there exists a positive hermitian matrix μ such that $\bar{\xi}'\mu\xi = \mu$ for all $\xi \in C$; it follows from the proof of existence of μ that μ can be chosen as an element of G. Then there exists $\eta \in G$ such that $\mu = \bar{\eta}'\eta$, and $\eta\xi\eta^{-1}$ is unitary. Let C_0 denote the group of all unitary elements of G; then C_0 is a compact subgroup of G, and any maximal compact subgroup of G is a conjugate subgroup $C = \eta^{-1}C_0\eta$, $\eta \in G$. Two points ξ_1, ξ_2 of G lie in the same right coset $C\xi_0$ of C, if and only if $\bar{\xi}_1'\mu\xi_1 = \bar{\xi}_2'\mu\xi_2$; consequently the topological space T of all positive hermitian matrices $\zeta = \bar{\xi}_0'\mu\xi_0$ in G is homeomorphic to the space $C\backslash G$. It is easily proved that the number of dimensions of T is $w_n - 1$, where $w_n = (hns + r_1 - r_3)ns/2$. In T the group G is represented by the transformations $\zeta \to \bar{\xi}'\zeta\xi$, $\xi \in G$. Let H be any discrete subgroup of the first kind in G; then the representation $\zeta \to \bar{\vartheta}'\zeta\vartheta$, $\vartheta \in H$, of H in T is discontinuous, and it follows from Theorem 1 that this representation is uniquely determined by the condition that the dimension of T be as small as possible.

15. Let an order J_1 in D_0 be given, and denote by J_n the order in A_0 consisting of all n-rowed matrices $\alpha = (\alpha_{kl})$, $\alpha_{kl} \in J_1$ ($k, l = 1, \cdots, n$). We assume now that the basis $\delta_1, \cdots, \delta_g$ of D_0 is a minimal basis of J_1; then an element of D lies in J_1, if and only if its coordinates are rational integers. We say that a column of n quantities $\xi_1, \cdots, \xi_n$ in D_0 is a *vector* $\mathfrak{x}$; if the *components* $\xi_1, \cdots, \xi_n$ lie in J_1, then $\mathfrak{x}$ is called *integral*.

Let $\zeta \in T$, and define $\bar{\mathfrak{x}}'\zeta\mathfrak{x} = \zeta[\mathfrak{x}]$, $\Phi(\mathfrak{x}) = \sigma(\zeta[\mathfrak{x}])$, where σ denotes the trace.

Obviously $\Phi(\mathfrak{x})$ is a positive quadratic form of gn variables, namely the coordinates of the components of the vector $\mathfrak{x}$. Let V be the volume of the ellipsoid $\Phi(\mathfrak{x}) < 1$, and determine $\eta \in G$ such that $\bar{\eta}'\eta = \zeta$; since the linear transformation $\eta\mathfrak{x} \to \mathfrak{x}$ has the determinant ± 1, it follows that V is independent of the point ζ in T. For our further purposes we do not need the exact value of V; a simple calculation, which we omit, leads to the formula

$$V = \pi^{\frac{gn}{2}} d^{-\frac{n}{2}} / \Gamma\left(1 + \frac{gn}{2}\right),$$

where d denotes the absolute value of the discriminant of the basis $\delta_1, \cdots, \delta_g$.

If $\mathfrak{x}_1, \cdots, \mathfrak{x}_k$ are vectors, then we denote by $L(\mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ the set of all vectors $\mathfrak{x}_1 t_1 + \cdots + \mathfrak{x}_k t_k$ with arbitrary rational $t_1, \cdots, t_k$. We determine gn integral vectors $\mathfrak{y}_k (k = 1, \cdots, gn)$ by the condition that the minimum of $\Phi(\mathfrak{x})$, in the set of all integral $\mathfrak{x}$ outside $L(\mathfrak{y}_1, \cdots, \mathfrak{y}_{k-1})$, be attained for $\mathfrak{x} = \mathfrak{y}_k$; if $k = 1$, the set $L(\mathfrak{y}_1, \cdots, \mathfrak{y}_{k-1})$ is defined as the null-vector. Put $\Phi(\mathfrak{y}_k) = N_k$, then $N_1 \leqq N_2 \leqq \cdots$ and, by Minkowski's theorem,

$$\prod_{k=1}^{gn} N_k \leqq 4^{gn} V^{-2}.$$

16. If $\mathfrak{x}_1, \cdots, \mathfrak{x}_k$ are vectors, then we denote by $L^*(\mathfrak{x}_1, \cdots, \mathfrak{x}_k)$ the set of all vectors $\mathfrak{x}_1\tau_1 + \cdots + \mathfrak{x}_k\tau_k$ with arbitrary $\tau_1, \cdots, \tau_k$ in D_0. We determine n integral vectors $\mathfrak{x}_k$ $(k = 1, \cdots, n)$ by the condition that the minimum of $\Phi(\mathfrak{x})$, in the set of all integral $\mathfrak{x}$ outside $L^*(\mathfrak{x}_1, \cdots, \mathfrak{x}_{k-1})$, be attained for $\mathfrak{x} = \mathfrak{x}_k$; if $k = 1$, the set $L^*(\mathfrak{x}_1, \cdots, \mathfrak{x}_{k-1})$ is defined as the null-vector. We call $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ an *extremal set*. Put $\Phi(\mathfrak{x}_k) = M_k$, then $M_1 \leqq \cdots \leqq M_n$.

In order to find a relation between these $\mathfrak{x}_k$ and the $\mathfrak{y}_k$ of the preceding section, we use an idea of Weyl. Let $\mathfrak{z}_1, \cdots, \mathfrak{z}_q$ be any vectors in the set $L^*(\mathfrak{x}_1, \cdots, \mathfrak{x}_{k-1})$, and let $q > g(k-1)$; then there exist q rational numbers $t_1, \cdots, t_q$, not all 0, such that $\mathfrak{z}_1 t_1 + \cdots + \mathfrak{z}_q t_q = 0$. Consequently, for any given positive index $k \leqq n$, there exists a uniquely determined positive index $l = l_k \leqq g(k-1) + 1 \leqq gn$, such that $\mathfrak{y}_t$ lies in $L^*(\mathfrak{x}_1, \cdots, \mathfrak{x}_{k-1})$ for $t < l$ and outside for $t = l$. By the definition of $\mathfrak{x}_k$, we obtain the inequality $\Phi(\mathfrak{y}_l) \geqq \Phi(\mathfrak{x}_k)$. On the other hand, the vector $\mathfrak{x}_k$ lies outside $L^*(\mathfrak{x}_1, \cdots, \mathfrak{x}_{k-1})$ and, *a fortiori*, outside $L(\mathfrak{y}_1, \cdots, \mathfrak{y}_{l-1})$; hence $\Phi(\mathfrak{x}_k) \geqq \Phi(\mathfrak{y}_l)$. Consequently, $\Phi(\mathfrak{x}_k) = \Phi(\mathfrak{y}_l)$,

$$M_k^g \leqq \prod_{r=g(k-1)+1}^{gk} N_r, \qquad \prod_{k=1}^{n} M_k \leqq 4^n V^{-\frac{2}{g}}.$$

17. Let $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ be an extremal set, and let $(\mathfrak{x}_1 \cdots \mathfrak{x}_n) = \nu = \nu_\zeta$; we have to bear in mind that ν_ζ is not necessarily uniquely determined by ζ. It follows from the construction of $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ that $\mathfrak{x}_1\tau_1 + \cdots + \mathfrak{x}_n\tau_n \neq 0$, for any system of quantities $\tau_1, \cdots, \tau_n$ in D_0, not all 0; consequently $|\nu| \neq 0$, and $|\nu|^2 = N$ is a positive rational integer, $N \geqq 1$.

Put $\mathfrak{x} = \nu\mathfrak{y}$, then $\zeta[\mathfrak{x}] = \rho[\mathfrak{y}]$, where $\rho = \bar{\nu}'\zeta\nu = \zeta[\nu]$ is the n-rowed matrix

with the elements $\rho_{kl} = \mathfrak{x}_k' \zeta \mathfrak{x}_l$ $(k, l = 1, \cdots, n)$ in D. Generalizing the Jacobi transformation, we have $\rho = \omega[\delta]$, where δ is a *triangular* matrix with the elements δ_{kl} $(k, l = 1, \cdots, n)$ in D, i.e., $\delta_{kl} = 0$ $(k > l)$ and $\delta_{kk} = \epsilon$, and ω is a diagonal matrix whose diagonal elements $\omega_1, \cdots, \omega_n$ lie in D. Let $\mathfrak{y} = (\eta_1, \cdots, \eta_n)'$, $\delta\mathfrak{y} = \mathfrak{z} = (\zeta_1 \cdots \zeta_n)'$, then

$$\zeta_k = \eta_k + \sum_{l=k+1}^{n} \delta_{kl} \eta_l \qquad (k = 1, \cdots, n).$$

Obviously the matrix $\omega_k (k = 1, \cdots, n)$ is positive hermitian. Since

$$\rho_{kk} = \omega_k + \sum_{l=1}^{k-1} \omega_l[\delta_{lk}] \qquad (k = 1, \cdots, n),$$

the hermitian matrix $\rho_{kk} - \omega_k$ is non-negative; hence

$$M_k = \Phi(\mathfrak{x}_k) = \sigma(\rho_{kk}) \geqq \sigma(\omega_k) > 0. \tag{1}$$

On the other hand,

$$N = |\nu|^2 = |\zeta[\nu]| = |\rho| = |\omega| = \prod_{k=1}^{n} |\omega_k|,$$

and consequently

$$N^{\frac{1}{g}} \prod_{k=1}^{n} \frac{\sigma(\omega_k)}{|\omega_k|^{\frac{1}{g}}} \prod_{k=1}^{n} \frac{M_k}{\sigma(\omega_k)} = \prod_{k=1}^{n} M_k \leqq 4^n V^{-\frac{2}{g}}. \tag{2}$$

Let $r_1, \cdots, r_g$ be the characteristic roots of the matrix ω_k; then

$$\sigma(\omega_k) = r_1 + \cdots + r_g \geqq g(r_1 \cdots r_g)^{\frac{1}{g}} = g\,|\omega_k|^{\frac{1}{g}}. \tag{3}$$

It follows from (1), (2), (3) that the numbers N, $\sigma(\omega_k)/|\omega_k|^{\frac{1}{g}}$, $M_k/\sigma(\omega_k)$ are bounded in T; then also the quotients r_p/r_q $(p, q = 1, \cdots, g)$ and, by the inequality $M_k \leqq M_{k+1}$, the quotients $\sigma(\omega_k)/\sigma(\omega_{k+1})$ $(k = 1, \cdots, n-1)$ are bounded in T.

Let k and l be given indices, $1 \leqq k < l \leqq n$, and choose $\eta_l = \epsilon$, $\eta_p = 0 (k < p \neq l)$, hence $\zeta_p = \delta_{pl}$ $(k < p < l)$, $\zeta_l = \epsilon$, $\zeta_p = 0$ $(p > l)$; furthermore, determine η_p $(p \leqq k)$ in J_1 by the condition that the coordinates of

$$\zeta_p = \eta_p + \sum_{q=p+1}^{n} \delta_{pq} \eta_q \qquad (p = k, k-1, \cdots, 1)$$

lie in the real interval $-\frac{1}{2} \leqq x < \frac{1}{2}$. Since

$$\mathfrak{x} = \nu\mathfrak{y} = \mathfrak{x}_1\eta_1 + \cdots + \mathfrak{x}_k\eta_k + \mathfrak{x}_l$$

is an integral vector outside $L^*(\mathfrak{x}_1, \cdots, \mathfrak{x}_{l-1})$, it follows that $\Phi(\mathfrak{x}) \geqq \Phi(\mathfrak{x}_l)$,

$$\sum_{p=1}^{k} \sigma(\omega_p[\zeta_p]) \geqq \sum_{p=1}^{k} \sigma(\omega_p[\delta_{pl}]);$$

but ζ_p $(p = 1, \cdots, k)$ is bounded, and consequently the quotients $\sigma(\omega_k[\delta_{kl}])/\sigma(\omega_k)$ $(1 \leqq k < l \leqq n)$ are bounded. Moreover there exists a unitary matrix u such that $\omega_k = \bar{u}'ru$, where r is the g-rowed diagonal matrix with the diagonal elements $r_1, \cdots, r_g$. It follows that δ_{kl} $(1 \leqq k < l \leqq n)$ is bounded in T.

18. Let $\mathfrak{P}$ be an m-rowed positive symmetric matrix with real elements. By the Jacobi transformation we have a uniquely determined representation $\mathfrak{P} = \mathfrak{Q}[\mathfrak{D}] = \mathfrak{D}'\mathfrak{Q}\mathfrak{D}$, where $\mathfrak{D} = (d_{kl})$ $(k, l = 1, \cdots, m)$ is a real triangular matrix, i.e., $d_{kl} = 0$ $(k > l)$ and $d_{kk} = 1$, and $\mathfrak{Q}$ is a diagonal matrix with positive diagonal elements $q_1, \cdots, q_m$. For any positive t, the $\frac{m(m+1)}{2} - 1$ inequalities

$$q_k/q_{k+1} < t, \qquad -t < d_{kl} < t \qquad (1 \leqq k < l \leqq m)$$

define an open set Π_t in the space of all $\mathfrak{P}$.

Consider now the positive hermitian matrix

$$\mathfrak{P}_\rho = (\bar{c}'\rho_{kl}c) = \omega[(\delta_{kl}c)] = \omega[(c\hat{\delta}_{kl})];$$

since $\bar{c}'\omega_k c = \bar{c}'c\hat{\omega}_k$ is real, the matrices $\bar{c}'\omega_k c$ and $\mathfrak{P}_\rho$ are real positive symmetric. It follows from the results of the preceding section that $\mathfrak{P}_\rho \in \Pi_f$, where the positive number f does not depend upon the point $\zeta = \rho[\nu^{-1}]$ in T.

Let H_0 be the group of all units ϑ in the order J_n; obviously H_0 is a discrete subgroup of G. The matrix $\nu = \nu_\zeta$ depends upon the point ζ in T. Since ν lies in J_n and $|\nu|$ is a bounded rational integer $\neq 0$, it follows that all ν lie in a finite number of right cosets of H_0 relative to the group xG, where x is any positive real number. Let $S = \nu_1, \cdots, \nu_p$ be a complete set of representatives of these cosets; then $\nu = \vartheta\nu_k$, $\nu_k \in S$, $\vartheta \in H_0$.

The point ζ of T is called *reduced*, whenever $\nu_\zeta \in S$; let Q be the set of all reduced ζ. The *images* $Q[\vartheta] = \bar{\vartheta}'Q\vartheta$, $\vartheta \in H_0$, cover the whole space T.

Let ζ be a reduced point, and let $\nu_\zeta = \nu \in S$. Since Π_f is open, there exists a neighborhood W_ζ of ζ in T such that $\mathfrak{P}_\rho \in \Pi_f$, for $\rho = \zeta_0[\nu]$ and all $\zeta_0 \in W_\zeta$. Let W_0 be the union of all these W_ζ, $\zeta \in Q$. Then W_0 is open, $Q \subset W_0$; moreover, for any point ζ_0 of W_0, there exists an element $\nu \in S$ such that $\mathfrak{P}_\rho \in \Pi_f$, for $\rho = \zeta_0[\nu]$. Assume now that also $\zeta_0[\vartheta^{-1}] = \zeta^* \in W_0$, for some $\vartheta \in H_0$, and determine $\nu^* \in S$ such that $\mathfrak{P}_{\rho^*} \in \Pi_f$, for $\rho^* = \zeta^*[\nu^*]$. Choose a positive rational integer a such that $a\nu_k^{-1} \in J_n$ $(k = 1, \cdots, p)$ and define $a\nu^{-1}\vartheta^{-1}\nu^* = \beta$, then $\beta \in J_n$, $|\beta| = \pm|a\nu^{-1}\nu^*|$, $\mathfrak{P}_\rho[\hat{\beta}] = a^2\mathfrak{P}_{\rho^*}$. But also $a^2\mathfrak{P}_{\rho^*}$ lies in Π_f, and $\hat{\beta}$ is a gn-rowed matrix with integral rational elements and bounded determinant. By a known theorem concerning the reduction of positive quadratic forms, it follows that β belongs to a finite set, independent of ζ; hence also ϑ belongs to a finite set. Consequently we see that $W_0[\vartheta] \cap W_0 \neq 0$ only for a finite number of $\vartheta \in H_0$.

19. In this section we shall prove that Q is contained in an open set of finite volume.

Let P be any open set in T, and let z be a positive number. We denote by $P(z)$ the set of all products $p = x\zeta$, where $0 < x < z$ and $\zeta \in P$. Let $v_z(P)$ be the euclidean volume of $P(z)$, computed in terms of the coordinates of the points $p \in P(z)$; obviously $v_z(P) = v_1(P)z^{a_n}$ with $a_n = w_n/gn$, where w_n is the number of dimensions of $P(z)$. For any $\xi \in G$, the linear transformation $p \to p[\xi]$ of the coordinates has the determinant 1; consequently $v_1(P)$ defines the invariant volume in T, up to a positive constant factor.

By the results of section 17, it is sufficient to prove the finiteness of $v_1(P_n)$, where P_n consists of all $\rho = \omega[\delta]$ in T satisfying

$$(4)\quad \sigma(\omega_k) < t \mid \omega_k \mid^{\frac{1}{\vartheta}} (k = 1, \cdots, n), \qquad \sigma(\omega_k)/\sigma(\omega_{k+1}) < t \quad (k = 1, \cdots, n-1),$$
$$\sigma(\bar{\delta}'_{kl}\delta_{kl}) < t \qquad (1 \leqq k < l \leqq n),$$

and t denotes an arbitrarily given positive number. The points of $P_n(1)$ are defined by (4) and the condition $\mid \rho \mid < 1$. Obviously the coordinates of $\omega_1 = \rho_{11}$ are bounded. We apply induction with respect to n. The assertion is trivial for $n = 1$, since then all coordinates of ρ are bounded. Let $n \geqq 2$, and let the assertion be proved for $n - 1$ instead of n.

We have

$$\rho[\mathfrak{y}] - \omega_1[\zeta_1] = \sum_{k=2}^{n} \omega_k[\zeta_k],$$

with $(\zeta_1 \cdots \zeta_n)' = \delta\mathfrak{y}$. For any fixed values of the coordinates of $\rho_{1k} = \omega_1\delta_{1k}$ $(k = 1, \cdots, n)$, let V_1 be the euclidean volume of the corresponding w_{n-1}-dimensional surface of section in $P_n(1)$. Since

$$\prod_{k=2}^{n} \mid \omega_k \mid = \mid \omega_1 \mid^{-1} \mid \rho \mid < \mid \omega_1 \mid^{-1},$$

we have

$$V_1 < v_1(P_{n-1}) \mid \omega_1 \mid^{-a_{n-1}}.$$

Instead of the coordinates of ρ_{1k} $(k = 1, \cdots, n)$ we introduce the coordinates of ω_1 and $\delta_{1k} = \omega_1^{-1}\rho_{1k}$ $(k = 2, \cdots, n)$, as variables of integration; these coordinates are bounded. Since the functional determinant of the transformation is $\mid \omega_1 \mid^{n-1}$, the integrand becomes

$$V_1 \mid \omega_1 \mid^{n-1} < v_1(P_{n-1}) \mid \omega_1 \mid^{n-a_{n-1}-1};$$

but $v_1(P_{n-1})$ is finite and

$$n - a_{n-1} = n - \frac{1}{2hs}(h(n-1)s + r_1 - r_3) =$$
$$\frac{n+1}{2} - \frac{r_1 - r_3}{2hs} \geqq \frac{n+1}{2} - \frac{1}{2s} \geqq \frac{n}{2} \geqq 1;$$

hence the integrand is bounded, and $v_1(P_n)$ is finite.

20. By the results of the two preceding sections, there exists in T an open set W of finite volume, such that the images $W[\vartheta]$, $\vartheta \in H_0$, cover T and that $W[\vartheta] \cap W \neq 0$ holds only for a finite number of $\vartheta \in H_0$. Let M be the set of all $\xi \in G$ such that $\mu[\xi] = \zeta \in W$; then M fulfills the conditions of Lemma 8. It follows that H_0 is a group of the first kind in G.

Let J be an arbitrary order in A_0, and let H be the group of units in J. Then $J^* = J \cap J_n$ is again an order, and the group H^* of units in J^* is a subgroup of finite indices j and j_0 in H and H_0. Let

$$H_0 = \sum_{k=1}^{j_0} a_k H^*, \qquad H = \sum_{l=1}^{j} H^* b_l, \qquad M_0 = \bigcup_k M a_k;$$

then

$$M_0 H = \bigcup_{k,l} M a_k H^* b_l = \bigcup_l M H_0 b_l = M H_0 = G,$$

and $v(M_0) \leqq j_0 v(M)$ is finite. If ξ is a common point of M_0 and $M_0\vartheta$, $\vartheta \in H$, then $\mu[\xi] = \zeta$ is a common point of $W[a_k]$ and $W[a_l\vartheta]$, for some k, $l \leqq j_0$. Exactly as in section 18, it can be proved that ϑ belongs to a finite set independent of ξ, if W is suitably chosen. Consequently M_0 satisfies the conditions of Lemma 8. It follows that H is a group of the first kind in G, and Theorem 2 is proved.

The explicit construction of a normal fundamental set in G relative to H_0, and more generally relative to H, is perhaps of minor interest; therefore we give only the following sketch: Let ζ_0 be a frontier point of Q; then it is easily seen, from the definition of Q, that there exist for ζ_0 two extremal sets $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ and $\mathfrak{y}_1, \cdots, \mathfrak{y}_n$, and an index $l \leqq n$, such that the differences $L_k(\zeta) = \Phi(\mathfrak{x}_k) - \Phi(\mathfrak{y}_k)$ vanish identically in ζ for $k = 1, \cdots, l-1$, but not for $k = l$, and that $L_l(\zeta_0) = 0$; moreover, $\mathfrak{x}_l$ and $\mathfrak{y}_l$ belong to a finite set independent of ζ_0; hence the frontier of Q lies on a finite number of planes. Let S be the set of all $\xi \in G$ such that $\mu[\xi] = \zeta$ is reduced; then the frontier of S lies on a finite number of surfaces of the second order; furthermore, if $S\vartheta \cap S \neq 0$, $\vartheta \in H_0$, then ϑ belongs to a finite set. Let F_0 be the set of all $\xi \in S$ such that $\sigma(\xi\vartheta) \geqq \sigma(\xi)$, whenever $\xi\vartheta \in S$ and $\vartheta \in H_0$; then the frontier of F_0 lies on a finite number of surfaces of the second order, and F_0 is a fundamental domain relative to H_0; i.e., the images $F_0\vartheta$, $\vartheta \in H_0$, cover G completely without overlappings, common frontier points excepted. Omitting a suitably selected set of frontier points of F_0, we obtain a normal fundamental set F relative to H_0, having only a finite number of neighbors. Finally, the passage from F to a fundamental set relative to H may easily be performed.

The group G is not connected; it consists of the two sets G_1 and G_2 defined by $|\xi| = 1$ and $|\xi| = -1$, each of them being connected. The group $G_1 \cap H$ is either H itself or an invariant subgroup of index 2. It follows from the result of section 9 that H has a finite system of generators.

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ON THE THEORY OF LOCAL RINGS

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The present paper contains a certain number of purely algebraic results to be used in a local theory of algebraic varieties, which will be developed in a later paper.

The most important part of this paper is contained in §IV, where is defined a notion of "multiplicity" (which might also have been called degree of ramification) in a complete local ring. This notion of multiplicity will appear eventually as the main tool in the definition of the multiplicity of intersection of two algebraic or algebroid varieties.

Conventions of terminology

By a ring we mean always a comutative ring with unit element. When a ring $\mathfrak{o}$ is said to be a sub-ring of a ring $\mathfrak{o}'$, it is understood tacitly that the unit element of $\mathfrak{o}$ is also the unit element of $\mathfrak{o}'$ (except when the contrary is explicitly specified). If $\mathfrak{o}$ is a sub-ring of $\mathfrak{o}'$ and if $\mathfrak{a}$ is an ideal in $\mathfrak{o}$, we denote by $\mathfrak{a}\mathfrak{o}'$ the ideal generated in $\mathfrak{o}'$ by the elements of $\mathfrak{a}$.

A ring $\mathfrak{o}$ is said to be *Noetherian* if the maximal condition holds for the ideals in $\mathfrak{o}$, i.e. if every ideal has a finite set of generators.

If $\mathfrak{a}$ is an ideal in a Noetherian ring $\mathfrak{o}$, we call "prime divisors" of $\mathfrak{a}$ the prime ideals which occur as associated prime ideals of the primary ideals in a shortest representation of $\mathfrak{a}$ as intersection of primary ideals.

We call "*adherence*" of a set E in a topological space the set which is usually called the closure of E. Any point of this set is said to be adherent to E.

The symbol $\varnothing$ denotes the empty set.

§I. Rings of Quotients[0]

It is well known that any ring $\mathfrak{o}$ can be imbedded in a "ring of quotients" whose elements are the fractions of the form a/b, $a \in \mathfrak{o}$, b not being a zero-divisor in $\mathfrak{o}$.

Let now S be any set of non-zero divisors in $\mathfrak{o}$ which is multiplicatively closed (i.e. the product of two elements of S is in S). Those elements of the ring of quotients which may be written in the form a/b, with $a \in \mathfrak{o}$, $b \in S$ form a sub-ring $\mathfrak{o}_S$ of the field of quotients, which we shall call the *ring of quotients* of the set S.

LEMMA 1. *If $\mathfrak{a}$ is an ideal in $\mathfrak{o}_S$, we have $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{o}) \cdot \mathfrak{o}_S$.*

The inclusion $(\mathfrak{a} \cap \mathfrak{o})\mathfrak{o}_S \subset \mathfrak{a}$ is obvious. Conversely, let a/b be any element of $\mathfrak{a}$, with $a \in \mathfrak{o}$, $b \in S$; we have $a = b.(a/b) \in \mathfrak{a} \cap \mathfrak{o}$, whence $a/b \in (\mathfrak{a} \cap \mathfrak{o})\mathfrak{o}_S$, which proves Lemma 1.

[0] The notion of ring of quotients of a set S is due to H. Grell (cf. *Beziehungen zwischen d. Idealen verschiedener Ringe*, Math. Ann. 97, 1926). The results contained in §I are not new (cf. Krull, *Idealtheorie*, Erg. d. Math.).

It follows immediately from Lemma 1 that $\mathfrak{o}_S$ is Noetherian when $\mathfrak{o}$ is itself Noetherian.

LEMMA 2. *Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$, and let $\mathfrak{q}$ be a primary ideal for $\mathfrak{p}$. If $\mathfrak{p} \cap S \neq \varnothing$, we have $\mathfrak{p}\mathfrak{o}_S = \mathfrak{q}\mathfrak{o}_S = \mathfrak{o}_S$. If $\mathfrak{p} \cap S = \varnothing$, the ideal $\mathfrak{p}\mathfrak{o}_S$ is prime in $\mathfrak{o}_S$; the ideal $\mathfrak{q}\mathfrak{o}_S$ is primary for $\mathfrak{p}\mathfrak{o}_S$; we have $\mathfrak{q}\mathfrak{o}_S \cap \mathfrak{o} = \mathfrak{q}$, and the primary ideals $\mathfrak{q}\mathfrak{o}_S$, $\mathfrak{q}$ have the same length.*

If $\mathfrak{p} \cap S \neq \varnothing$, we have also $\mathfrak{q} \cap S \neq 0$ (because any power of an element of S is in S), whence $\mathfrak{p}\mathfrak{o}_S = \mathfrak{q}\mathfrak{o}_S = \mathfrak{o}_S$. Assume now that $\mathfrak{p} \cap S = \varnothing$; then 1 cannot belong to $\mathfrak{p}\mathfrak{o}_S$, whence $\mathfrak{p}\mathfrak{o}_S \neq \mathfrak{o}_S$[1] and, *a fortiori*, $\mathfrak{q}\mathfrak{o}_S \neq \mathfrak{o}_S$. Let a/b, a'/b' be two elements of $\mathfrak{o}_S$ such that $a/b \cdot a'/b' \in \mathfrak{q}\mathfrak{o}_S$, $a'/b' \notin \mathfrak{q}\mathfrak{o}_S$ (with $a, a' \in \mathfrak{o}$, $b', b \in S$, $a' \notin \mathfrak{q}$). We have $aa'/bb' = c/c'$, $c \in \mathfrak{q}$, $c' \in S$, whence $c'aa' = cbb' \in \mathfrak{q}$; since $\mathfrak{p} \cap S = \varnothing$, c' does not belong to $\mathfrak{p}$, whence $aa' \in \mathfrak{q}$, and therefore $a^r \in \mathfrak{q}$ for some r; it follows that $(a/b)^r \in \mathfrak{q}\mathfrak{o}_S$, which proves that $\mathfrak{q}\mathfrak{o}_S$ is primary. If $\mathfrak{q} = \mathfrak{p}$, we can furthermore take $r = 1$, which proves that $\mathfrak{p}\mathfrak{o}_S$ is prime.

We have $\mathfrak{q}\mathfrak{o}_S \subset \mathfrak{p}\mathfrak{o}_S$; conversely, if $a/b \in \mathfrak{p}\mathfrak{o}_S$ ($a \in \mathfrak{p}$, $b \in S$), we have $a^r \in \mathfrak{q}$ for some r, whence $(a/b)^r \in \mathfrak{q}\mathfrak{o}_S$, which proves that $\mathfrak{q}\mathfrak{o}_S$ is primary for $\mathfrak{p}\mathfrak{o}_S$.

Let a be an element of $\mathfrak{q}\mathfrak{o}_S \cap \mathfrak{o}$; we may write a in the form b/c, $b \in \mathfrak{q}$, $c \in S$. Since $c \notin \mathfrak{p}$, the relation $ac \in \mathfrak{q}$ implies $a \in \mathfrak{q}$, whence $\mathfrak{q}\mathfrak{o}_S \cap \mathfrak{o} = \mathfrak{q}$. It follows that there exists a one-to-one inclusion preserving correspondence between the primary ideals for $\mathfrak{p}$ in $\mathfrak{o}$ and the primary ideals for $\mathfrak{p}\mathfrak{o}_S$ in $\mathfrak{o}_S$, which proves that $\mathfrak{q}$ and $\mathfrak{q}\mathfrak{o}_S$ have the same length.

LEMMA 3. *Assume that $\mathfrak{o}$ is Noetherian; let $\mathfrak{a} = (\bigcap_i \mathfrak{q}_i) \bigcap (\bigcap_j \mathfrak{v}_j)$ be an irredundant[2] representation of $\mathfrak{a}$ as an intersection of primary ideals, where we have separated the primary ideals $\mathfrak{q}_i$ with prime associated ideals $\mathfrak{p}_i$ such that $\mathfrak{p}_i \cap S = \varnothing$ from those $\mathfrak{v}_j$ whose associated prime ideals meet the set S. Then $\mathfrak{a}\mathfrak{o}_S = \bigcap_i \mathfrak{q}_i\mathfrak{o}_S$, and this is an irredundant representation of $\mathfrak{a}\mathfrak{o}_S$ as intersection of primary ideals in $\mathfrak{o}_S$.*

We have obviously $\mathfrak{a}\mathfrak{o}_S \subset \bigcap_i \mathfrak{q}_i\mathfrak{o}_S$; let conversely a be an element of $\bigcap_i \mathfrak{q}_i\mathfrak{o}_S$; for each i we may write $a = b_i/c_i$, $b_i \in \mathfrak{q}_i$, $c_i \in S$. For each index j corresponding to a $\mathfrak{v}_j$, we can find an element c_j in $\mathfrak{v}_j \cap S$; we set $c = \prod_i c_i \cdot \prod_j c_j$; we have $c \in S$ and $a = b/c$, with $b \in (\bigcap_i \mathfrak{q}_i) \bigcap (\bigcap_j \mathfrak{v}_j)$ whence $b \in \mathfrak{a}$ and $a \in \mathfrak{a}\mathfrak{o}_S$. In order to prove that the representation $\mathfrak{a}\mathfrak{o}_S = \bigcap_i \mathfrak{q}_i\mathfrak{o}_S$ is irredundant, we observe first that it follows from Lemma 2 that the prime divisors associated with the primary ideals $\mathfrak{q}_i$ are distinct from each other; moreover an inclusion $\mathfrak{q}_i\mathfrak{o}_S \subset \bigcap_{i' \neq i} \mathfrak{q}_{i'}\mathfrak{o}_S$ would imply by Lemma 1 that $\mathfrak{q}_i \subset \bigcap_{i' \neq i} \mathfrak{q}_{i'}$: this is impossible.

Let us now consider the case where $\mathfrak{o}$ does not contain any zero-divisor $\neq 0$ and where S is the set of the elements of $\mathfrak{o}$ which do not belong to some given prime ideal $\mathfrak{p}$. This set is clearly multiplicatively closed; the corresponding ring $\mathfrak{o}_S$ takes then the name of "quotient ring" of the ideal $\mathfrak{p}$ with respect to the ring $\mathfrak{o}$; we shall denote this ring by $\mathfrak{o}_\mathfrak{p}$. It is clear that the factor ring $\mathfrak{o}_\mathfrak{p}/\mathfrak{p}\mathfrak{o}_\mathfrak{p}$ is the field of quotients of $\mathfrak{o}/\mathfrak{p}$.

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[1] Observe that, if $\mathfrak{a}$ is any ideal in $\mathfrak{r}$, the elements of $\mathfrak{a}\mathfrak{o}_\mathfrak{r}$ are of the form ax, $a \in \mathfrak{a}$, $x \in \mathfrak{r}$.

[2] This means that none of the ideals $\mathfrak{a}_i$, $\mathfrak{a}_j$ is contained in the intersection of the others.

Let us return to the general case, where S is any multiplicatively closed set of regular elements of a ring $\mathfrak{o}$. If $\mathfrak{p}$ is a prime ideal in $\mathfrak{o}$ which does not meet S, it follows from Lemma 2 that $\mathfrak{o}/\mathfrak{p}$ may be identified with a sub-ring of $\mathfrak{o}_S/\mathfrak{p}\mathfrak{o}_S$; it is clear that $\mathfrak{o}_S/\mathfrak{p}\mathfrak{o}_S$ has the same field of quotients as $\mathfrak{o}/\mathfrak{p}$; but, in general, $\mathfrak{o}_S/\mathfrak{p}\mathfrak{o}_S$ will not be itself a field.

§II. Semi-local Rings

DEFINITION 1. *A Noetherian ring $\mathfrak{o}$ is called a semi-local ring if there exist only a finite number of maximal prime ideals in $\mathfrak{o}$.*

Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ be all the distinct maximal prime ideals in a semi-local ring $\mathfrak{o}$. We set $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_h$. We shall define a topology in the ring $\mathfrak{o}$; a subset U of $\mathfrak{o}$ will be called an open set in this topology if it satisfies the following condition: if $u \in U$, there exists an exponent n (depending upon u) such that $u + \mathfrak{a}^n \subset U$. It is obvious that any union or finite intersection of open sets in the sense of this definition are open; the empty set and $\mathfrak{o}$ itself are open. The usual requirements for the definition of a topology are therefore satisfied. If u is any element of $\mathfrak{o}$, the sets $u + \mathfrak{a}^n$ form a fundamental system of neighbourhoods of u; if we observe that $(u + \mathfrak{a}^n) - (v + \mathfrak{a}^n) = u - v + \mathfrak{a}^n$, $(u + \mathfrak{a}^n)(v + \mathfrak{a}^n) \subset uv + \mathfrak{a}^n$, we see that the operations of addition, subtraction, multiplication in $\mathfrak{o}$ are continuous. Whenever we shall apply topological notions to a semi-local ring, it will be tacitly understood that these notions apply to the topology we have just defined.

We shall now prove that the Hausdorf separation axiom holds in our topology. It will be sufficient to prove that $\bigcap_{n=1}^{\infty}\mathfrak{a}^n = \{0\}$.

LEMMA 1. *If $\mathfrak{o}$ is a semi-local ring, an element of $\mathfrak{o}$ which does not belong to any one of the maximal prime ideals in $\mathfrak{o}$ is a unit in $\mathfrak{o}$.*

In fact, if we had $\mathfrak{o}u \neq \mathfrak{o}$, $\mathfrak{o}u$ would be contained in some prime ideal of $\mathfrak{o}$, and therefore also in some maximal prime ideal, which is not the case.

Taking Lemma 1 into account, the formula $\bigcap_{n=1}^{\infty}\mathfrak{a}^n = \{0\}$ follows immediately from

LEMMA 2. *Let $\mathfrak{a}$ be an ideal in a Noetherian ring $\mathfrak{o}$. The following statements are equivalent:* 1) *no element which is $\equiv 1 \pmod{\mathfrak{a}}$ is a zero divisor in $\mathfrak{o}$;* 2) *we have $\bigcap_{n=1}^{\infty}\mathfrak{a}^n = \{0\}$.*[3]

Let $\mathfrak{n}$ be the ideal $\bigcap_{n=1}^{\infty}\mathfrak{a}^n$. We shall prove that $\mathfrak{n} = \mathfrak{n}\mathfrak{a}$. Let $\mathfrak{n}\mathfrak{a} = \bigcap_i \mathfrak{q}_i$ be a representation of $\mathfrak{n}\mathfrak{a}$ as an intersection of primary ideals, and let $\mathfrak{p}_i$ be the prime divisor of $\mathfrak{q}_i$. If $\mathfrak{a} \not\subset \mathfrak{p}_i$, we have $\mathfrak{q}_i : \mathfrak{a} = \mathfrak{q}_i$; in fact, let a be an element of $\mathfrak{a}$ not contained in $\mathfrak{p}_i$, and let q be an element of $\mathfrak{q}_i : \mathfrak{a}$; we have $aq \in \mathfrak{q}_i$, $a \notin \mathfrak{p}_i$, whence $q \in \mathfrak{q}_i$. Since $\mathfrak{n}\mathfrak{a} \subset \mathfrak{q}_i$, we have $\mathfrak{n} \subset \mathfrak{q}_i$ if $\mathfrak{a} \not\subset \mathfrak{p}_i$. This last conclusion holds also if $\mathfrak{a} \subset \mathfrak{p}_i$; in fact, we have $\mathfrak{p}_i^k \subset \mathfrak{q}_i$ for some k, whence $\mathfrak{n} \subset \mathfrak{a}^k \subset \mathfrak{p}_i^k \subset \mathfrak{q}_i$. It follows that $\mathfrak{n} \subset \bigcap_i \mathfrak{q}_i = \mathfrak{n}\mathfrak{a}$, whence $\mathfrak{n} = \mathfrak{n}\mathfrak{a}$. Let us now take a set of generators $\nu_1, \cdots, \nu_r$ of $\mathfrak{n}$; we have $\nu_i = \sum_{j=1}^{r} a_{ij}\nu_j$ with $a_{ij} \in \mathfrak{a}$; the determinant δ of the matrix $(\delta_{ij} - a_{ij})$ is $\equiv 1 \pmod{\mathfrak{a}}$ and we have $\delta\nu_i = 0$ $(1 \leqq i \leqq r)$; there-

[3] Cf. *Krull*, Dimensionstheorie in Stellenringen, J.f.d.r.u.M. 179, 1938, pp. 204–226.

fore, if 1) holds, we have $v_i = 0$ $(1 \leqq i \leqq r)$, whence $\mathfrak{n} = \{0\}$. Conversely, let us assume that there exist two elements a, b such that $a \in \mathfrak{a}$, $b \neq 0$, $(1 + a)b = 0$; we have $b = -ab = (-1)^n a^n b \in \mathfrak{a}^n$ for every n, whence $b \in \mathfrak{n}$, $\mathfrak{n} \neq \{0\}$.

Since the topology of $\mathfrak{o}$ is a Hausdorf topology, we may use the notion of limit. Since the first countability axiom obviously holds in $\mathfrak{o}$, it will be sufficient to consider limits of sequences. We shall say that a sequence (u_n) is "convergent" if $\lim_{n\to\infty}(u_{n+1} - u_n) = 0$, i.e. if, given any $k > 0$, we have $u_{n+1} - u_n \in \mathfrak{a}^k$ as soon as n is sufficiently large.

DEFINITION 2. *The semi-local ring $\mathfrak{o}$ is said to be complete if every convergent sequence of elements of $\mathfrak{o}$ has a limit in $\mathfrak{o}$.*

LEMMA 3. *Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ be the maximal prime ideals in a semi-local ring $\mathfrak{o}$. Let $a_1, \cdots, a_h$ be h given elements of $\mathfrak{o}$, and let n be a given exponent. The system of congruences $x \equiv a_i \pmod{\mathfrak{p}_i^n}$ has always a solution x, which is uniquely determined modulo $(\mathfrak{p}_1 \cdots \mathfrak{p}_h)^n$.*

It is clear that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $j \neq i$. Let $\mathfrak{a}_i$ be the ideal $\prod_{j\neq i}\mathfrak{p}_j$; we have $\mathfrak{a}_i \not\subset \mathfrak{p}_i$, whence $\mathfrak{a}_i + \mathfrak{p}_i = \mathfrak{o}$, since $\mathfrak{p}_i$ is a maximal prime ideal. It follows that there exists an element e_i of $\mathfrak{a}_i$ which is $\equiv 1 \pmod{\mathfrak{p}_i}$. Set $e_{i,n} = 1 - (1 - e_i^n)^n$; we have $e_{i,n} \in \mathfrak{a}_i^n$, $e_{i,n} \equiv 1 \pmod{\mathfrak{p}_i^n}$. The element $x = \sum_{i=1}^h a_i e_{i,n}$ is a solution of our system of congruences. If x' is any other solution, we have $(x' - x)\sum_{i=1}^h e_{i,n} \equiv 0 \pmod{(\mathfrak{p}_1 \cdots \mathfrak{p}_h)^n}$; by Lemma 1, $\sum_{i=1}^h e_{i,n}$ is a unit in $\mathfrak{o}$, whence $x' - x \equiv 0 \pmod{(\mathfrak{p}_1 \cdots \mathfrak{p}_h)^n}$.

PROPOSITION 1. *Let $\mathfrak{o}$ be a semi-local ring. If $\mathfrak{b}$ is an ideal $\neq \mathfrak{o}$ in $\mathfrak{o}$, $\mathfrak{o}/\mathfrak{b}$ is a semi-local ring. If $\mathfrak{o}$ is complete, $\mathfrak{o}/\mathfrak{b}$ is complete.*

Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_{h_1}$ be the maximal prime ideals of $\mathfrak{o}$ which contain $\mathfrak{b}$, and let $\mathfrak{p}_{h_1+1}, \cdots, \mathfrak{p}_h$ be the other maximal prime ideals of $\mathfrak{o}$. The maximal prime ideals of $\mathfrak{o}/\mathfrak{b}$ are clearly the ideals $\mathfrak{p}_i/\mathfrak{b}$ $(1 \leqq i \leqq h_1)$, which proves that $\mathfrak{o}/\mathfrak{b}$ is semi-local. Assume that $\mathfrak{o}$ is complete, and let (u_n^*) be a convergent sequence of elements of $\mathfrak{o}/\mathfrak{b}$. The element $v_n^* = u_{n+1}^* - u_n^*$ is the residue class of an element v_n of $\mathfrak{o}$ which belongs to $(\mathfrak{p}_1 \cdots \mathfrak{p}_{h_1})^{m(n)}$, with $\lim_{n\to\infty} m(n) = \infty$. If $j > h_1$, we have $\mathfrak{b} + \mathfrak{p}_j = \mathfrak{o}$; it follows that $\mathfrak{p}_j$ contains an element $\overline{w}_j$ which is $\equiv 1 \pmod{\mathfrak{b}}$. We set $v_n' = v_n \cdot (\prod_{j>h_1}\overline{w}_j)^{m(n)}$; let u_1 be an element whose residue class modulo $\mathfrak{b}$ is u_1^*; we set $u_n = u_1 + \sum_{k=1}^{n-1} v_k'$; u_n is a representative of the residue class u_n^*, and the sequence (u_n) is convergent in $\mathfrak{o}$. Let u be its limit; it is clear that the sequence (u_n^*) converges in $\mathfrak{o}/\mathfrak{b}$ to the residue class u^* of u.

PROPOSITION 2. *Let $\mathfrak{o}$ be a complete local ring, and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ be the distinct maximal prime ideals of $\mathfrak{o}$. To every $\mathfrak{p}_i$ $(1 \leqq i \leqq h)$ there is associated an idempotent ϵ_i in $\mathfrak{o}$, with the following properties: if $\mathfrak{o}\epsilon_i$ is considered as a ring with the unit element ϵ_i, it is a complete semi-local ring whose only maximal prime ideal ideal is $\mathfrak{p}_i\epsilon_i$; we have $\epsilon_i \in \mathfrak{p}_j$ for $j \neq i$; the idempotents $\epsilon_1, \cdots, \epsilon_h$ are mutually orthogonal and their sum is 1.*

We make use of the elements $e_{i,n}$ which were constructed in the proof of Lemma 3. It follows immediately from Lemma 3 that $e_{i,n+1} - e_{i,n} \in (\mathfrak{p}_1 \cdots \mathfrak{p}_h)^n$; it follows that each one of the h sequences $(e_{i,n})$ is convergent, and has therefore

a limit ϵ_i in $\mathfrak{o}$. We have $e_{i,n}e_{j,n} \in (\mathfrak{p}_1 \cdots \mathfrak{p}_h)^n$ (for $i \neq j$), $1 - \sum_{i=1}^{h} e_{i,n} \in (\mathfrak{p}_1 \cdots \mathfrak{p}_h)^n$, whence $\epsilon_i\epsilon_j = 0$ for $i \neq j$ and $1 = \sum_{i=1}^{h} \epsilon_i$. Since $\epsilon_i = \lim_{n\to\infty} e_{i,n}$, $\epsilon_i - e_{i,n}$ is contained in $\mathfrak{p}_1 \cdots \mathfrak{p}_h$, and, *a fortiori*, in $\mathfrak{p}_j$ if n is large enough; if $j \neq i$, we have also $e_{i,n} \in \mathfrak{p}_j$, whence $\epsilon_i \in \mathfrak{p}_j$. Th ring $\mathfrak{o}\epsilon_i$ is isomorphic to $\mathfrak{o}/\mathfrak{o}(1 - \epsilon_i)$; Proposition 2 follows therefore from Proposition 1.

LEMMA 4.[4] *Let $\mathfrak{o}$ be a Noetherian ring which does not contain any zero divisor $\neq 0$ and let $\mathfrak{o}'$ be a ring which contains $\mathfrak{o}$ as a sub-ring and is a finite $\mathfrak{o}$-module. If $\mathfrak{a}$ is an ideal $\neq \mathfrak{o}$ in $\mathfrak{o}$, we have $\mathfrak{a}\mathfrak{o}' \neq \mathfrak{o}'$.*

Let us say that m elements $x_1, \cdots, x_m$ of $\mathfrak{o}'$ are linearly independent over $\mathfrak{o}$ if the conditions $\sum_{i=1}^{m} a_i x_i = 0$, $a_i \in \mathfrak{o}$ imply $a_i = 0$ $(1 \leqq i \leqq m)$. If $\mathfrak{o}' = \sum_{j=1}^{M} \mathfrak{o} y_j$, with $y_1 = 1$, we may extract from the set $\{y_1, \cdots, y_M\}$ a set of linearly independent elements $\{x_1, \cdots, x_m\}$ containing $x_1 = y_1 = 1$ and which is not properly contained in any other larger sub-set of $\{y_1, \cdots, y_M\}$ composed of linearly independent elements. It follows that there exists an element $c \neq 0$ in $\mathfrak{o}$ such that $c\mathfrak{o}' \subset \sum_{i=1}^{m} \mathfrak{o} x_i$, whence $c\mathfrak{a}^n\mathfrak{o}' \subset \sum_{i=1}^{m} \mathfrak{a}^n x_i$. Assume for a moment that $\mathfrak{a}\mathfrak{o}' = \mathfrak{o}'$, whence $1 \in \mathfrak{a}^n\mathfrak{o}'$ for every n. It follows that $c = \sum_{i=1}^{m} c_{i,n}x_i$ with $c_{i,n} \in \mathfrak{a}^n$ $(1 \leqq i \leqq m)$; since $x_1, \cdots, x_m$ are linearly independent over $\mathfrak{o}$ and $x_1 = 1$, we have $c = c_{1,n} \in \mathfrak{a}^n$ for all n, in contradiction with Lemma 1.

REMARK. The same argument proves that $\mathfrak{a}^n\mathfrak{o}' \cap \mathfrak{o} \subset \mathfrak{a}^n{:}\mathfrak{o}c$; we shall use this remark later.

LEMMA 5. *The result of Lemma 4 still holds if $\mathfrak{o}$ contains zero divisors.*

It is sufficient to consider the case where $\mathfrak{a}$ is a prime ideal $\mathfrak{p}$. The ideal $\mathfrak{p}$ contains some minimal prime divisor $\mathfrak{n}$ of the zero ideal. Let $\{0\} = \bigcap_i \mathfrak{v}_i$ be an irredundant representation of the zero ideal as an intersection of primary ideals, $\mathfrak{v}_1$ being the component of $\mathfrak{n}$; we have $\mathfrak{v}_1{:}\mathfrak{n} \neq \mathfrak{v}_1$, whence $(\mathfrak{v}_1{:}\mathfrak{n}) \cap (\bigcap_{i>1} \mathfrak{v}_i) \neq \{0\}$. If c is an element $\neq 0$ in the latter ideal, we have $c\mathfrak{n} = \{0\}$, $c \notin \mathfrak{v}_1$. Let $\mathfrak{n}'$ be the ideal $\mathfrak{n}\mathfrak{o}' \cap \mathfrak{o}$: we have $c\mathfrak{n}' = \{0\}$; if $\mathfrak{n}'$ were to contain an element ν' not contained in $\mathfrak{n}$, the equality $c\nu' = 0$ would imply $c \in \mathfrak{v}_1$, which is not the case. We have therefore $\mathfrak{n}' = \mathfrak{n}$. Since $\mathfrak{o}'$ is a finite $\mathfrak{o}$-module, $\mathfrak{o}'/\mathfrak{n}\mathfrak{o}'$ is a finite module over $\mathfrak{o}/\mathfrak{n}$. By Lemma 4, the ideal generated by $\mathfrak{p}/\mathfrak{n}$ in $\mathfrak{o}'/\mathfrak{n}\mathfrak{o}'$ is $\neq \mathfrak{o}'/\mathfrak{n}\mathfrak{o}'$, whence $\mathfrak{p}\mathfrak{o}' \neq \mathfrak{o}'$.

PROPOSITION 3. *Let $\mathfrak{o}$ be a semi-local ring, and let $\mathfrak{o}'$ be a ring which contains $\mathfrak{o}$ as a sub-ring and is a finite $\mathfrak{o}$-module. Then $\mathfrak{o}'$ is a semi-local ring. If $\mathfrak{o}$ is complete $\mathfrak{o}'$ is also complete.*

Let $\mathfrak{p}'$ be a maximal prime ideal in $\mathfrak{o}'$; if $\mathfrak{p} = \mathfrak{p}' \cap \mathfrak{o}$, the ring $\mathfrak{o}'/\mathfrak{p}'$ contains $\mathfrak{o}/\mathfrak{p}$ over which it is a finite module. But $\mathfrak{o}'/\mathfrak{p}'$ is a field; it follows immediately from Lemma 4 that $\mathfrak{o}/\mathfrak{p}$ cannot contain any ideal besides $\{0\}$ and $\mathfrak{o}/\mathfrak{p}$ itself; this means that $\mathfrak{o}/\mathfrak{p}$ is a field, i.e. that $\mathfrak{p}$ is a maximal prime ideal in $\mathfrak{o}$. Moreover, $\mathfrak{o}'/\mathfrak{p}\mathfrak{o}'$ is a finite module over $\mathfrak{o}/\mathfrak{p}$, i.e. is a hypercomplex system over $\mathfrak{o}/\mathfrak{p}$; it follows that there are only a finite number of prime ideals in $\mathfrak{o}'$ which contain $\mathfrak{p}$. We have therefore proved that $\mathfrak{o}'$ is a semi-local ring. Let $\mathfrak{a}$ be the product

[4] Lemma 4 is a generalization of a theorem of Krull (cf. Beitrage zur Arithmetik kommutativer Integritatsberiche, M. Z. 42, 1937, Satz 1, p. 749).

of the maximal prime ideals of $\mathfrak{o}$; every prime ideal in $\mathfrak{o}'$ which contains $\mathfrak{a}$ is maximal in $\mathfrak{o}'$, which proves that $\mathfrak{a}\mathfrak{o}'$ contains some power of the product $\mathfrak{a}'$ of all maximal prime ideals of $\mathfrak{o}'$. Assume now that $\mathfrak{o}$ is complete, and let (u'_n) be a convergent sequence of elements of $\mathfrak{o}'$; it follows from what we have said that $v'_n = u'_{n+1} - u'_n \in \mathfrak{a}^{m(n)}\mathfrak{o}'$, with $\lim_{n\to\infty} m(n) = \infty$. Let $y_1, \cdots, y_h$ be elements of $\mathfrak{o}'$ such that $\mathfrak{o}' = \sum_{j=1}^{h} \mathfrak{o} y_j$; we may write $v'_n = \sum_{i=1}^{h} v_{n,i} y_i$ with $v_{n,i} \in \mathfrak{a}^{m(n)}$. We set also $u'_1 = \sum_{i=1}^{h} u_{1,i} y_i$, $u_{n,i} = u_{1,i} + \sum_{k=1}^{n-1} v_{k,i}$. Each one of the sequences $(u_{n,i})$ is convergent in $\mathfrak{o}$; if we set $u_i = \lim_{n\to\infty} u_{n,i}$, the element $\sum_{i=1}^{h} u_i y_i - u'_n$ belongs to a power of $\mathfrak{a}\mathfrak{o}'$ whose exponent increases indefinitely with n, which proves that the sequence (u'_n) converges in $\mathfrak{o}'$ to the limit $\sum_{i=1}^{h} u_i y_i$.

LEMMA 6.[5]. *Let $\mathfrak{o}$ be a semi-local ring, and let $\mathfrak{b}$ be an ideal in $\mathfrak{o}$. If $\mathfrak{a}$ is the product of the maximal prime ideals of $\mathfrak{o}$, we have $\bigcap_{n=1}^{\infty}(\mathfrak{b} + \mathfrak{a}^n) = \mathfrak{b}$.*

We know that $\mathfrak{o}/\mathfrak{b}$ is a semi-local ring and that $\mathfrak{a} + \mathfrak{b}/\mathfrak{b}$ is contained in the product of the maximal prime ideals of $\mathfrak{o}/\mathfrak{b}$ (cf. the proof of Proposition 1). It follows that $\bigcap_{n=1}^{\infty}(\mathfrak{a} + \mathfrak{b}/\mathfrak{b})^n = \bigcap_{n=1}^{\infty}(\mathfrak{a}^n + \mathfrak{b}/\mathfrak{b} = \{0\}$, which proves Lemma 6.

LEMMA 7. *Let $\mathfrak{o}$ be a complete semi-local ring, and let $(\mathfrak{b}_n)$ be a sequence of ideals in $\mathfrak{o}$ such that $\mathfrak{b}_{n+1} \subset \mathfrak{b}_n$ $(1 \leqq n < \infty)$ and $\bigcap_{n=1}^{\infty} \mathfrak{b}_n = \{0\}$. If $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ are the maximal prime ideals of $\mathfrak{o}$, we have $\mathfrak{b}_n \subset (\mathfrak{p}_1 \cdots \mathfrak{p}_h)^{m(n)}$, where $m(n)$ is an exponent which increases indefinitely with n.*

We set $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_h$; it follows immediately from Lemma 3 that $\mathfrak{o}/\mathfrak{a}^n$ is isomorphic to the direct product of the rings $\mathfrak{o}/\mathfrak{p}_i^n$ $(1 \leqq i \leqq h)$; since $\mathfrak{p}_i$ is the only prime ideal to contain $\mathfrak{p}_i^n$, $\mathfrak{p}_i^n$ is a primary ideal. It is well known that the descending chain condition for ideals holds in $\mathfrak{o}/\mathfrak{p}_i^n$; therefore it holds also in $\mathfrak{o}/\mathfrak{a}^n$. This being said, let h be any fixed integer; we set $\mathfrak{b}'_h = \bigcap_{n=1}^{\infty}(\mathfrak{a}_h + \mathfrak{b}_n)$. It follows from what we have just said that there exists an integer $N(h)$ such that $\mathfrak{a}^h + \mathfrak{b}_n = \mathfrak{b}'_h$ for every $n \geqq N(h)$; therefore we have $\mathfrak{b}'_h = \mathfrak{a}^h + \mathfrak{b}'_{h+1}$! Let b_h be any given element of $\mathfrak{b}'_h$; we define by induction an element $b_{h+n} \in \mathfrak{b}'_{h+n}$ in the following way: if b_{h+n} is already defined, we represent it in the form $a_{h+n} + b_{h+n+1}$, with $a_{h+n} \in \mathfrak{a}^{h+n}$, $b_{h+n+1} \in \mathfrak{b}'_{h+n+1}$, and this defines b_{h+n+1}. Our construction shows that the sequence (b_{h+n}) is convergent; let b be its limit. We have $b_{h+n} \equiv b_h \pmod{\mathfrak{a}^h}$ for every n, whence $b \equiv b_h \pmod{\mathfrak{a}^h}$; on the other hand, we have also $b \equiv b_{h+n} \pmod{\mathfrak{a}^{h+n}}$, and $b_{h+n} \in \bigcap_{m=1}^{\infty}(\mathfrak{a}^{h+n} + \mathfrak{b}_m)$, whence $b \in \mathfrak{a}^{h+n} + \mathfrak{b}_m$ for every m and n. By Lemma 6, we have $b \in \mathfrak{b}_m$ for every m, whence $b = 0$, $b_h \in \mathfrak{a}^h$, $\mathfrak{b}'_h = \mathfrak{a}^h$, $\mathfrak{b}_n \subset \mathfrak{a}^h$ for $n \geqq N(h)$, which proves Lemma 7.

PROPOSITION 4. *Let $\mathfrak{o}$ be a complete semi-local ring; we denote by $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ the maximal prime ideals of $\mathfrak{o}$ and by $\mathfrak{a}$ the ideal $\mathfrak{p}_1 \cdots \mathfrak{p}_h$. Let $\mathfrak{o}'$ be a ring which contains $\mathfrak{o}$ as a sub-ring. If $\bigcap_{n=1}^{\infty} \mathfrak{a}^n\mathfrak{o}' = \{0\}$, we have $\mathfrak{a}\mathfrak{o}' \cap \mathfrak{o} = \mathfrak{a}$, $\mathfrak{a}^n\mathfrak{o}' \cap \mathfrak{o} \subset \mathfrak{a}^{m(n)}$, where $m(n)$ is an exponent which increases indefinitely with n. If we assume furthermore that $\mathfrak{o}'/\mathfrak{a}\mathfrak{o}'$ is a finite module over $\mathfrak{o}/\mathfrak{a}$, $\mathfrak{o}'$ is a complete semi-local ring and is a finite $\mathfrak{o}$-module. Moreover, $\mathfrak{o}$ is a sub-space of $\mathfrak{o}'$.*

It follows immediately from Lemma 7 that $\mathfrak{a}^n\mathfrak{o}' \cap \mathfrak{o} \subset \mathfrak{a}^{m(n)}$, with $\lim_{n\to\infty}$

[5] This Lemma is due to Krull, Satz 2, l.c. note 3), p. 692.

$m(n) = \infty$. Therefore each $\mathfrak{p}_i$ contains $\mathfrak{a}^n\mathfrak{o}' \cap \mathfrak{o}$, and, a *fortiori*, $(\mathfrak{a}\mathfrak{o}' \cap \mathfrak{o})^n$ when n is sufficiently large; since $\mathfrak{p}_i$ is prime, we have $\mathfrak{a}\mathfrak{o}' \cap \mathfrak{o} \subset \bigcap_{i=1}^h \mathfrak{p}_i$. It follows immediately from Lemma 3 that $\bigcap_{i=1}^h \mathfrak{p}_i = \mathfrak{a}$, whence $\mathfrak{a}\mathfrak{o}' \cap \mathfrak{o} = \mathfrak{a}$. Assuming that $\mathfrak{o}'/\mathfrak{a}\mathfrak{o}'$ is a finite module over $\mathfrak{o}/\mathfrak{a}$, let $v_1, \cdots, v_d$ be elements of $\mathfrak{o}'$ such that $\mathfrak{o}'/\mathfrak{a}\mathfrak{o}' = \sum_{i=1}^d \mathfrak{o}/\mathfrak{a}.\ v_i^*$, where v_i^* is the residue class of v_i (mod. $\mathfrak{a}\mathfrak{o}'$). Every element of $\mathfrak{o}'$ is congruent (mod. $\mathfrak{a}\mathfrak{o}'$) to an element of $\sum_{i=1}^d \mathfrak{o}v_i$. Let also $\{a_1, \cdots, a_m\}$ be a set of generators of the ideal $\mathfrak{a}$, and let x be any element of $\mathfrak{o}'$; we construct by induction d sequences $(x_{i,n})_{n=0,1,\ldots}$ of elements of $\mathfrak{o}$ such that $x \equiv \sum_{i=1}^m x_{i,n}v_i$ (mod. $\mathfrak{a}^n\mathfrak{o}'$). We set $x_{i,0} = 0$; assume that the elements $x_{i,n}$ have already been constructed; let $\xi_1, \cdots, \xi_N$ be all elements of $\mathfrak{o}$ which may be expressed as monomials of degree n in $a_1, \cdots, a_m$; they form a set of generators of $\mathfrak{a}^n\mathfrak{o}'$, whence $x - \sum_{i=1}^d x_{i,n}v_i = \sum_{k=1}^N y_k\xi_k$ with $y_k \in \mathfrak{o}'$; we have $y_k \equiv \sum_{i=1}^d y_{ki}v_i$ (mod. $\mathfrak{a}\mathfrak{o}'$) with $y_{ki} \in \mathfrak{o}$: we may take $x_{i,n+1} = x_{i,n} + \sum_{k=1}^N y_{ki}\xi_k$. At the same time, we see that each of the sequences $(x_{i,n})$ is convergent in $\mathfrak{o}$, and has therefore a limit x_i. We set $x' = \sum_{i=1}^d x_iv_i$; since $x_i \equiv x_{i,n}$ (mod. $\mathfrak{a}^n$), we have $x' - x \in \mathfrak{a}^n\mathfrak{o}'$ for every n, whence $x' = x$. We have proved that $\mathfrak{o}'$ is a finite $\mathfrak{o}$-module. Proposition 4 follows therefore from Proposition 3 above.

Let $\mathfrak{o}$ be a complete semi-local ring. Let A be any (finite or infinite) set of indices, and assume that we have assigned to every $\alpha \in A$ an element $u_\alpha \in \mathfrak{o}$, in such a way that, given any $k > 0$, we have $u_\alpha \in \mathfrak{a}^k$ for almost all α (i.e. for all but a finite number), where $\mathfrak{a}$ is the product of the maximal ideals of $\mathfrak{o}$. Let v_k be the sum of those elements u_α which are not in $\mathfrak{a}^k$; the sequence (v_k) has a limit v in $\mathfrak{o}$, and we shall set $v = \sum_{\alpha \in A} u_\alpha$. Such sums are associative; i.e., if A is represented as the union of a family of mutually disjoint sub-sets B_β, we have $\sum_{\alpha \in A} u_\alpha = \sum_\beta (\sum_{\alpha \in B_\beta} u_\alpha)$.

Let $\mathfrak{r}$ be a sub-ring of $\mathfrak{o}$; we shall denote by $\mathfrak{r}[[X_1, \cdots, X_m]]$ the ring of the formal power series in m letters $X_1, \cdots, X_m$ with coefficients in $\mathfrak{r}$. If F is any such power series, we may write $F = \sum_0^\infty F_k$, where F_k is a form of degree k. Let $a_1, \cdots, a_m$ be any m elements of $\mathfrak{a}$; the infinite sum $\sum_0^\infty F_k(a_1, \cdots, a_m)$ has a meaning in $\mathfrak{o}$; we shall set $F(a_1, \cdots, a_m) = \sum_0^\infty F_k(a_1, \cdots, a_m)$. It is clear that the mapping $F \to F(a_1, \cdots, a_m)$ is a homomorphism of $\mathfrak{r}[[X_1, \cdots, X_m]]$ into $\mathfrak{o}$. If this homomorphism turns out to be an isomorphism, we shall say that $a_1, \cdots, a_m$ are *analytically independent* over $\mathfrak{r}$.

We shall now prove that any semi-local ring can be imbedded in a certain complete semi-local ring. First, we prove

LEMMA 8. *Let $\mathfrak{o}$ be a Noetherian ring. The ring $\mathfrak{o}[[X_1, \cdots, X_r]]$ of the formal power series in r letters $X_1, \cdots, X_r$ with coefficients in $\mathfrak{o}$ is a Noetherian ring.*

Let $\mathfrak{x}$ be the ideal generated by $X_1, \cdots, X_r$. The elements of $\mathfrak{x}^n$ are the power series which do not contain any term of total degree $< n$. It follows immediately that $\bigcap_{n=1}^\infty \mathfrak{x}^n = \{0\}$. If F is any power-series $\neq 0$, and if $F \in \mathfrak{x}^d$, $F \notin \mathfrak{x}^{d+1}$ (with the convention that $\mathfrak{x}^0$ is the whole ring), we can find a homogeneous form F^* of degree d such that $F - F^* \in \mathfrak{x}^{d+1}$, and F^* is uniquely determined by this condition: F^* is called the beginning form of F. Let $\mathfrak{A}$ be an ideal $\neq \{0\}$ in $\mathfrak{o}[[X_1, \cdots, X_r]]$; the beginning forms of the elements $\neq 0$ in $\mathfrak{A}$

generate a homogeneous ideal $\mathfrak{A}^*$ in the ring $\mathfrak{o}[X_1, \cdots, X_r]$. The latter ring being Noetherian, $\mathfrak{A}^*$ has a finite set of generators $A_1^*, \cdots, A_m^*$; we may assume that each A_i^* is the beginning form of an element $A_i \in \mathfrak{A}$. Let d_i be the degree of the form A_i^*; if A^* is a form $\neq 0$ of degree d in $\mathfrak{A}^*$, we may write A^* in the form $\sum_i B_i A_i^*$, where each B_i is a form of degree $d - d_i$ ($B_i = 0$ if $d < d_i$); it follows immediately that A^* is the beginning form of the element $\sum_i B_i A_i \in \mathfrak{A}$.

Let C be any element of $\mathfrak{A}$; we shall define by induction m sequences $(C_{i,n})_{n=0,1,\ldots}$ of forms in the following way: we set $C_{i,0} = 0$ $(1 \leqq i \leqq m)$; if the elements $C_{i,k}$ have already been defined for $k < n$, we construct the element $C - \sum_{k=0}^{n-1} \sum_{i=0}^{m} C_{i,k} A_i$; if this element is 0, we set $C_{i,n} = 0$ for $1 \leqq i \leqq m$; if not, it has a beginning form of degree say e_n, and we may write this beginning form in the form $\sum_{i=1}^{m} C_{i,n} A_i^*$, where each $C_{i,n}$ is a form of degree $e_n - d_i$ ($C_{i,n} = 0$ if $d_i > e_n$). If it happens for some n that all $C_{i,n}$ are 0, A clearly belongs to the ideal generated by $A_1, \cdots, A_m$ in $\mathfrak{o}[[X_1, \cdots, X_r]]$; if not, we have $e_{n+1} > e_n$ for all n, whence $\lim e_n = \infty$. It follows that $\sum_{n=0}^{\infty} C_{i,n}$ is a power series C_i and we have clearly $C = \sum_i A_i C_i$, which proves that $\mathfrak{A}$ is the ideal generated by $A_1, \cdots, A_m$. This shows that $\mathfrak{o}[[X_1, \cdots, X_r]]$ is a Noetherian ring.

Let (F_n) be any convergent sequence of elements of our ring such that $F_{n+1} - F_n$ belongs to a power of $\mathfrak{x}$ whose exponent increases indefinitely with n. If M is any monomial in $X_1, \cdots X_r$, the coefficient $a(M; F_n)$ of M in F_n remains constant when n is sufficiently large; let $a(M)$ be this constant value. The sum $\sum_M a(M)M$, extended over all monomials, represents a power series F; we shall set $F = \sum_1^{\infty} F_n$.

Let now $\mathfrak{o}$ be any semi-local ring; we denote by $\mathfrak{a}$ the product of the maximal prime ideals of $\mathfrak{o}$ and by $\{a_1, \cdots, a_m\}$ a set of generators of $\mathfrak{a}$. We construct with m letters $X_1, \cdots, X_m$ the ring $\mathfrak{o}[[X_1, \cdots, X_m]]$ of the power series with coefficients in $\mathfrak{o}$. If $F = \sum_0^{\infty} F_k$ is an element of this ring (with F_k homogeneous of degree k), we set $u_n(F) = \sum_{k=0}^{n} F_k(a_1, \cdots, a_m)$, and we denote by $\mathfrak{n}$ the set of the power-series F for which we have $\lim_{n\to\infty} u_n(F) = 0$. We see at once that $\mathfrak{n}$ is an ideal in $\mathfrak{o}[[X_1, \cdots, X_m]]$. We set $\bar{\mathfrak{o}} = \mathfrak{o}[[X_1, \cdots, X_m]]/\mathfrak{n}$. The ring $\mathfrak{o}$ is a sub-ring of $\mathfrak{o}[[X_1, \cdots, X_m]]$ and has obviously only 0 in common with $\mathfrak{n}$; we may therefore identify $\mathfrak{o}$ with the sub-ring of $\bar{\mathfrak{o}}$ upon which it is mapped isomorphically in the natural mapping of $\mathfrak{o}[[X_1, \cdots, X_m]]$ onto $\bar{\mathfrak{o}}$. Let $\mathfrak{x}$ be the ideal generated by $X_1, \cdots, X_m$ in $\mathfrak{o}[[X_1, \cdots, X_m]]$, and let $\bar{\mathfrak{a}}$ be the ideal $\mathfrak{x} + \mathfrak{n}/\mathfrak{n}$ in $\bar{\mathfrak{o}}$. We have clearly $X_i \equiv a_i \pmod{\mathfrak{n}}$ $(1 \leqq i \leqq m)$, whence $\bar{\mathfrak{a}} = \mathfrak{a}\bar{\mathfrak{o}}$. We shall prove that $\bar{\mathfrak{a}}^n \cap \mathfrak{o} = \mathfrak{a}^n$. Let y be any element of $\bar{\mathfrak{a}}^n \cap \mathfrak{o}$; y is the residue class modulo $\mathfrak{n}$ of a power series F such that $F \in \mathfrak{x}^n$, $F - y \in \mathfrak{n}$; since $\lim_{k\to\infty} (u_k(F) - y) = 0$, there exists an index $N > n$ such that $y \equiv u_N(F) \pmod{\mathfrak{a}^n}$; since $F \in \mathfrak{x}^n$, we have $u_N(F) = \sum_{k=n}^{N} F_k(a_1, \cdots, a_m) \in \mathfrak{a}^n$, whence $y \in \mathfrak{a}^n$, which proves our assertion.

It is clear that, if $u \in \bar{\mathfrak{o}}$ and n is given > 0, there always exists an element $u' \in \mathfrak{o}$ such that $u - u' \in \bar{\mathfrak{a}}^n$. Since $\bar{\mathfrak{a}} \cap \mathfrak{o} = \mathfrak{a}$, it follows that $\bar{\mathfrak{o}}/\bar{\mathfrak{a}}$ is isomorphic with $\mathfrak{o}/\mathfrak{a}$. If $\mathfrak{p}$ is any maximal prime ideal in $\mathfrak{o}$, we have $\bar{\mathfrak{a}} \subset \mathfrak{p}\bar{\mathfrak{o}}$, and $\bar{\mathfrak{o}}/\mathfrak{p}\bar{\mathfrak{o}}$ is

isomorphic with $\mathfrak{o}/\mathfrak{p}$; it follows that $\mathfrak{p}\bar{\mathfrak{o}}$ is a maximal prime ideal in $\bar{\mathfrak{o}}$. Any element u of $\bar{\mathfrak{o}}$ which is $\equiv 1 \pmod{\bar{\mathfrak{a}}}$ is a unit; in fact, u is the residue class (mod. $\mathfrak{n}$) of a power series of the form $1 + X$, with $X \in \mathfrak{x}$; such a series has an inverse in $\mathfrak{o}[[X_1, \cdots, X_m]]$, which can be represented by the series $1 - X + X^2 - \cdots + (-1)^n X^n + \cdots$; it follows that u has an inverse in $\mathfrak{o}$. If $\bar{\mathfrak{p}}$ is any maximal prime ideal in $\mathfrak{o}$, we have therefore $1 \notin \bar{\mathfrak{p}} + \bar{\mathfrak{a}}$, whence $\bar{\mathfrak{a}} \subset \bar{\mathfrak{p}}$, and it follows that $\bar{\mathfrak{p}}$ is of the form $\mathfrak{p}\bar{\mathfrak{o}}$, $\mathfrak{p}$ being some maximal prime ideal in $\mathfrak{o}$. We know that $\bar{\mathfrak{o}}$ is Noetherian by Lemma 8; it follows that $\bar{\mathfrak{o}}$ is a semi-local ring. If (u_n) is a sequence of elements of $\bar{\mathfrak{o}}$ which is convergent in $\bar{\mathfrak{o}}$, we have $u_{n+1} - u_n \in \bar{\mathfrak{a}}^{m(n)}$, with $\lim_{n\to\infty} m(n) = \infty$. It follows that $u_{n+1} - u_n$ is the residue class modulo $\mathfrak{n}$ of a power series $V_n \in \mathfrak{x}^{m(n)}$. The sum $\sum_1^\infty V_n$ represents a power series V; if v is the residue class of V, we have clearly $u_1 + v = \lim_{n\to\infty} u_n$, which proves that $\bar{\mathfrak{o}}$ is complete.

Let now $\mathfrak{o}^*$ be a complete semi-local ring which contains $\mathfrak{o}$ as a sub-ring and is such that $\mathfrak{a}$ is contained in the product $\mathfrak{a}^*$ of the maximal prime ideals of $\mathfrak{o}^*$. Let F be any element of $\mathfrak{o}[[X_1, \cdots, X_m]]$; the sequence $(u_n(F))$ is convergent in $\mathfrak{o}$, and therefore also in $\mathfrak{o}^*$; as such it has a limit $\theta_1(F)$ in $\mathfrak{o}^*$; θ_1 is clearly a homomorphism of $\mathfrak{o}[[X_1, \cdots, X_m]]$ into $\mathfrak{o}^*$ and maps $\mathfrak{n}$ onto $\{0\}$. It follows that θ_1 defines in a natural way a homomorphism θ of $\bar{\mathfrak{o}}$ into $\mathfrak{o}^*$ which maps every element of $\mathfrak{o}$ upon itself. The ring $\theta(\bar{\mathfrak{o}})$ is contained in the adherence of $\mathfrak{o}$, considered as a sub-set of $\mathfrak{o}^*$; on the other hand, $\theta(\bar{\mathfrak{o}})$ is a complete semi-local ring; it follows immediately from Lemma 7 that $\theta(\bar{\mathfrak{o}})$ is a sub-space of $\mathfrak{o}^*$. Since $\theta(\bar{\mathfrak{o}})$ is complete, it must be a closed sub-space of $\mathfrak{o}^*$. $\theta(\bar{\mathfrak{o}})$ is therefore the adherence of $\mathfrak{o}$ in $\mathfrak{o}^*$.

If $\mathfrak{o}$ is everywhere dense in $\mathfrak{o}^*$, we have $\theta(\bar{\mathfrak{o}}) = \mathfrak{o}^*$. We may observe also that, in this case, the condition $\mathfrak{a} \subset \mathfrak{a}^*$ is automatically verified. In fact, $\mathfrak{o}^*/\mathfrak{a}^*$ is isomorphic to $\mathfrak{o}/\mathfrak{a}^* \cap \mathfrak{o}$, which proves that every prime ideal of $\mathfrak{o}$ containing $\mathfrak{a}^* \cap \mathfrak{o}$ is maximal in $\mathfrak{o}$, and that $\mathfrak{a}^* \cap \mathfrak{o}$ is the product of these prime ideals, whence $\mathfrak{a} \subset \mathfrak{a}^* \cap \mathfrak{o}$.

If $\mathfrak{o}$ is a sub-space of $\mathfrak{o}^*$, it is clear that the elements of $\mathfrak{n}$ are the only ones to be mapped on 0 under θ; in this case, θ is therefore an isomorphism. We have proved

THEOREM 1.[6] *Let $\mathfrak{o}$ be a semi-local ring. There exists a complete semi-local ring $\mathfrak{o}$ containing $\bar{\mathfrak{o}}$ as a sub-ring and a sub-space in which $\bar{\mathfrak{o}}$ is everywhere dense. If $\mathfrak{o}_1$ is another ring with the same properties, there exists an isomorphism of $^-$ with $\mathfrak{o}_1$ which maps every element of $\mathfrak{o}$ upon itself. If $\mathfrak{o}^*$ is any complete local ring containing $\mathfrak{o}$ as a sub-ring, and if the product of the maximal prime ideals of $\mathfrak{o}$ is contained in the product of the maximal prime ideals of $\mathfrak{o}^*$, the adherence of $\mathfrak{o}$ in $\mathfrak{o}^*$ is a complete local ring which is a homomorphic image of $\bar{\mathfrak{o}}$. If $\mathfrak{o}$ is a sub-space of $\mathfrak{o}^*$, the adherence of $\mathfrak{o}$ in $\mathfrak{o}^*$ is isomorphic with $\bar{\mathfrak{o}}$.*

DEFINITION 3. *The ring $\bar{\mathfrak{o}}$ which is described in Theorem 1 is called the completion of the semi-local ring $\mathfrak{o}$.*

[6] Theorem 1 is a precision of Satz 14 of Krull, l.c. note 3), p. 692.

PROPOSITION 5. *If $\mathfrak{b}$ is an ideal in a semi-local ring $\mathfrak{o}$, and if $\bar{\mathfrak{o}}$ is the completion of $\mathfrak{o}$, we have $\mathfrak{b}\bar{\mathfrak{o}} \cap \mathfrak{o} = \mathfrak{b}$. The ring $\bar{\mathfrak{o}}/\mathfrak{b}\bar{\mathfrak{o}}$ is the completion of $\mathfrak{o}/\mathfrak{b}$.*

Let $\{b_1, \cdots, b_m)\}$ be a set of generators of $\mathfrak{b}$; if $u \in \mathfrak{b}\bar{\mathfrak{o}} \cap \mathfrak{o}$, we have $u = \sum_{i=1}^{m} b_i u_i$, $u_i \in \bar{\mathfrak{o}}$; if n is any given exponent, we can find elements $u_{i,n} \in \mathfrak{o}$ such that $u_i \equiv u_{i,n} \pmod{\mathfrak{a}^n\bar{\mathfrak{o}}}$, where $\mathfrak{a}$ is the product of the maximal prime ideals of $\mathfrak{o}$. We have $b - \sum_i b_i u_{i,n} \in \mathfrak{a}^n\bar{\mathfrak{o}} \cap \mathfrak{o} = \mathfrak{a}^n$ (cf. the proof of Theorem 1), and $\sum_i b_i u_{i,n} \in \mathfrak{b}$, whence $u \in \mathfrak{b} + \mathfrak{a}^n$; by Lemma 6, we have $u \in \mathfrak{b}$, whence $\mathfrak{b}\bar{\mathfrak{o}} \cap \mathfrak{o} = \mathfrak{b}$. It follows that $\bar{\mathfrak{o}}/\mathfrak{b}\bar{\mathfrak{o}}$ contains $\mathfrak{o}/\mathfrak{b}$ as a sub-ring; this sub-ring is clearly everywhere dense. The product of the maximal prime ideals of $\mathfrak{o}/\mathfrak{b}$ is $\mathfrak{a} + \mathfrak{b}/\mathfrak{b}$; the product of the maximal prime ideals of $\bar{\mathfrak{o}}/\mathfrak{b}\bar{\mathfrak{o}}$ is $\mathfrak{a} + \mathfrak{b}\bar{\mathfrak{o}}/\mathfrak{b}\bar{\mathfrak{o}}$. Let u^* be any element of $(\mathfrak{a} + \mathfrak{b}\bar{\mathfrak{o}}/\mathfrak{b}\bar{\mathfrak{o}})^n \cap \mathfrak{o}/\mathfrak{b}$; u^* is the residue class modulo $\mathfrak{b}\bar{\mathfrak{o}}$ of an element $u \in \mathfrak{a}^n\bar{\mathfrak{o}}$ and is also the residue class modulo $\mathfrak{b}$ of an element $u' \in \mathfrak{o}$. Since $\mathfrak{b}\bar{\mathfrak{o}} \subset \mathfrak{b} + \mathfrak{a}^n\bar{\mathfrak{b}}$, there exists an element $b \in \mathfrak{b}$ such that $u' - b \in \mathfrak{a}^n\bar{\mathfrak{o}} \cap \mathfrak{o} = \mathfrak{a}^n$ whence $u^* \in (\mathfrak{a} + \mathfrak{b}/\mathfrak{b})^n$. We have proved that $(\mathfrak{a} + \mathfrak{b}\bar{\mathfrak{o}}/\mathfrak{b}\bar{\mathfrak{o}})^n \cap \mathfrak{o}/\mathfrak{b} = (\mathfrak{a} + \mathfrak{b}/\mathfrak{b})^n$, which shows that $\mathfrak{o}/\mathfrak{b}$ is a sub-space of $\bar{\mathfrak{o}}/\mathfrak{b}\bar{\mathfrak{o}}$: Proposition 5 is proved.

PROPOSITION 6. *Let $\bar{\mathfrak{o}}$ be the completion of a semi-local ring $\mathfrak{o}$. If an element $u \in \mathfrak{o}$ is not a zero divisor in $\mathfrak{o}$, it is not a zero divisor in $\bar{\mathfrak{o}}$.*

Assume that $uv = 0$, $v \in \bar{\mathfrak{o}}$. If $\mathfrak{a}$ is the product of the maximal prime ideals in $\mathfrak{o}$, we can find for every $n > 0$ an element $v_n \in \mathfrak{o}$ such that $v - v_n \in \mathfrak{a}^n\bar{\mathfrak{o}}$. We have $uv_n \in u\mathfrak{a}^n\bar{\mathfrak{o}} \cap \mathfrak{o} = u\mathfrak{a}^n$. Since u is not a zero divisor in $\mathfrak{o}$, it follows that $v_n \in \mathfrak{a}^n$, whence $v = 0$, which proves Proposition 6.

LEMMA 9. *Let $\mathfrak{a}$ be the product of the maximal prime ideals in a semi-local ring $\mathfrak{o}$, and let c be an element of $\mathfrak{o}$ which is not a zero divisor. Then $\mathfrak{a}^n{:}\mathfrak{o}c$ is contained in a power of $\mathfrak{a}$ whose exponent increases indefinitely with n.*

Let $\bar{\mathfrak{o}}$ be the completion of $\mathfrak{o}$. If $u \in \mathfrak{a}^n\bar{\mathfrak{o}}{:}\bar{\mathfrak{o}}c$ for very $n > 0$, we have $cu \in \bigcap_{n=1}^{\infty} \mathfrak{a}^n\bar{\mathfrak{o}} = \{0\}$, whence $u = 0$ by Proposition 6. By Lemma 7, we have $\mathfrak{a}^n\bar{\mathfrak{o}}{:}\bar{\mathfrak{o}}c \subset \mathfrak{a}^{m(n)}\bar{\mathfrak{o}}$, with $\lim_{n\to\infty} m(n) = \infty$. Lemma 9 follows if we observe that $\mathfrak{a}^n{:}\mathfrak{o}c \subset \mathfrak{a}^n\bar{\mathfrak{o}}{:}\bar{\mathfrak{o}}c \cap \mathfrak{o}$.

PROPOSITION 7. *Let $\mathfrak{o}$, $\mathfrak{o}'$ be two semi-local rings on which we make the following assumptions: $\mathfrak{o}'$ contains $\mathfrak{o}$ as a sub-ring and is a finite $\mathfrak{o}$-module; no element $\neq 0$ in $\mathfrak{o}$ is a zero divisor in $\mathfrak{o}'$. Let $\bar{\mathfrak{o}}$, $\bar{\mathfrak{o}}'$ be the completions of $\mathfrak{o}$, $\mathfrak{o}'$ respectively. Then $\bar{\mathfrak{o}}$ is a sub-ring of $\bar{\mathfrak{o}}'$ (up to an isomorphism); if $\mathfrak{o}' = \sum_{i=1}^{m} \mathfrak{o}y_i$, we have $\bar{\mathfrak{o}}' = \sum_{i=1}^{m} \bar{\mathfrak{o}}y_i$. If m elements $x_1, \cdots, x_m$ of $\mathfrak{o}'$ are linearly independent over $\mathfrak{o}$, they are also linearly independent over $\bar{\mathfrak{o}}$. If an element of $\bar{\mathfrak{o}}$ is a zero divisor in $\bar{\mathfrak{o}}'$, it is already a zero divisor in $\bar{\mathfrak{o}}$.*

It follows from Lemma 9 and from the remark which follows the proof of Lemma 4 that $\mathfrak{o}$ is a sub-space of $\mathfrak{o}'$, and therefore also of $\bar{\mathfrak{o}}'$. By Theorem 1, we may assume that $\bar{\mathfrak{o}} \subset \bar{\mathfrak{o}}'$. Set $\bar{\mathfrak{o}}_1' = \sum_{i=1}^{m} \mathfrak{o}y_i$; $\bar{\mathfrak{o}}_1'$ is a finite $\bar{\mathfrak{o}}$-module, and is therefore a complete semi-local ring. If $\mathfrak{a}$ is the product of the maximal prime ideals of $\mathfrak{o}$, $\mathfrak{a}\bar{\mathfrak{o}}'$ is contained in every maximal prime ideal of $\mathfrak{o}'$, and therefore $\mathfrak{a}\bar{\mathfrak{o}}'$ is contained in every maximal prime ideal of $\bar{\mathfrak{o}}'$ whence $\bigcap_{n=1}^{\infty} \mathfrak{a}^n\bar{\mathfrak{o}}' = \{0\}$. It follows from Lemma 7 that $\bar{\mathfrak{o}}_1'$ is a sub-space of $\bar{\mathfrak{o}}'$; since $\bar{\mathfrak{o}}_1'$ is complete, it is a closed sub-set of $\bar{\mathfrak{o}}'$; since $\mathfrak{o}' \subset \bar{\mathfrak{o}}_1'$, we have $\bar{\mathfrak{o}}' = \bar{\mathfrak{o}}_1'$.

If the elements $x_1, \cdots, x_m$ are linearly independent over $\mathfrak{o}$, we may extend

the set $\{x_1, \cdots, x_m\}$ to a maximal set of linearly independent elements (for instance by adjoining some of the elements y_j). We may therefore assume without loss of generality that the set $\{x_1, \cdots x_m\}$ is already a maximal set of linearly independent elements. It follows that there exists an element $c \neq 0$ in $\mathfrak{o}$ such that $c\mathfrak{o}' \subset \sum_{i=1}^m \mathfrak{o}x_i$. This being said, assume that $\sum_{i=1}^m u_i x_i = 0$ $(u_i \in \bar{\mathfrak{o}})$. For every $n > 0$,we can find elements $u_{i,n} \in \mathfrak{o}$ such that $u_i - u_{i,n} \in \mathfrak{a}^n\bar{\mathfrak{o}}$, whence $\sum_{i=1}^n u_{i,n}x_i \in \sum_{i=1}^n \mathfrak{a}^n\bar{\mathfrak{o}}'x_i \cap \mathfrak{o}' \subset \mathfrak{a}^n\mathfrak{o}'$. It follows that $\sum_{i=1}^m cu_{i,n}x_i \in \sum_{i=1}^m \mathfrak{a}^n x_i$, $\sum_{i=1}^m cu_{i,n}x_i = \sum_{i=1}^m a_{i,n}x_i$, $a_{i,n} \in \mathfrak{a}^n$. The elements x_i being linearly independent over $\mathfrak{o}$, we have $cu_{i,n} = a_{i,n} \in \mathfrak{a}^n$, $u_{i,n} \in \mathfrak{a}^n{:}\mathfrak{o}c \subset \mathfrak{a}^{m(n)}$ with $\lim_{n\to\infty} m(n) = \infty$ (by Lemma 9). It follows that $u_i = \lim_{n\to\infty} u_{i,n} = 0$, which proves that $x_1, \cdots, x_m$ are linearly independent over $\bar{\mathfrak{o}}$.

Let u be an element of $\bar{\mathfrak{o}}$, and assume that $uv = 0$, $v \in \bar{\mathfrak{o}}'$. Using the same notations as above, we may write $cv = \sum_{i=1}^m u_i x_i$, $u_i \in \bar{\mathfrak{o}}$, whence $\sum_{i=1}^m uu_i x_i = 0$, $uu_i = 0$ $(1 \leqq i \leqq m)$. If u is not a zero divisor in $\bar{\mathfrak{o}}$, we have $u_i = 0$ $(i \leqq i \leqq m)$, whence $cv = 0$ and $v = 0$ by Proposition 6.

PROPOSITION 8. *Let $\mathfrak{o}$ be a semi-local ring which does not contain any zero divisor $\neq 0$, and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ be the maximal prime ideals of $\mathfrak{o}$. Let $\mathfrak{o}_i$ be the quotient ring of $\mathfrak{p}_i$ with respect to $\mathfrak{o}$. The completion of $\mathfrak{o}$ is isomorphic to the direct product of the completions of the rings $\mathfrak{o}_i$.*

We use, for the ring $\bar{\mathfrak{o}}$, the notations of Proposition 2. It follows from Proposition 6 that $\mathfrak{o}\epsilon_i$ is a ring which is isomorphic with $\mathfrak{o}$. Let a be an element of $\mathfrak{o}$ which does not belong to $\mathfrak{p}_i$; since $\mathfrak{p}_i$ is maximal in $\mathfrak{o}$, there exists an element $b \in \mathfrak{o}$ such that $ab \equiv 1$ (mod. $\mathfrak{p}_i$); it follows that $ab\epsilon_i$ has an inverse in $\bar{\mathfrak{o}}\epsilon_i$; this means that $\bar{\mathfrak{o}}\epsilon_i$ contains the quotient ring $\mathfrak{o}_i\epsilon_i$ of $\mathfrak{p}_i\epsilon_i$ with respect to $\mathfrak{o}\epsilon_i$. This quotient ring is clearly everywhere dense in $\bar{\mathfrak{o}}\epsilon_i$. Let a be an element of $\mathfrak{o}$ such that $a\epsilon_i \in \mathfrak{p}_i^n\bar{\mathfrak{o}}\epsilon_i$; we have $a = a\epsilon_i + a(1 - \epsilon_i) \in \mathfrak{p}_i^n\bar{\mathfrak{o}}$, since $1 - \epsilon_i \in \mathfrak{p}_i^n\bar{\mathfrak{o}}$ for every n; it follows that $a\epsilon_i \in \mathfrak{p}_i^n\epsilon_i$. The ideal $\mathfrak{p}_i^n\bar{\mathfrak{o}} \cap \mathfrak{o}_i\epsilon_i$ is an ideal of $\mathfrak{o}_i\epsilon_i$ whose intersection with $\mathfrak{o}\epsilon_i$ is $\mathfrak{p}_i^n\epsilon_i$; by Lemma 2, §I, p. 691, this ideal is $\mathfrak{p}_i^n\mathfrak{o}_i\epsilon_i$, which proves that $\mathfrak{o}_i\epsilon_i$ is a sub-space of $\bar{\mathfrak{o}}\epsilon_i$; $\bar{\mathfrak{o}}\epsilon_i$ is therefore the completion of $\mathfrak{o}_i\epsilon_i$, which proves Proposition 8.

§III. Local Rings

The notion of local ring has been introduced by Krull.[7] He has defined a local ring as being a Noetherian ring in which the non-units form an ideal. The local rings which we shall consider in this paper are however of a more restricted type. Before defining them, we make the following observation: if a ring $\mathfrak{o}$ contains a field K as a sub-field, and if $\mathfrak{m}$ is an ideal in $\mathfrak{o}$, the field K is mapped isomorphically under the natural mapping of $\mathfrak{o}$ onto $\mathfrak{o}/\mathfrak{m}$, and therefore K may be identified with a sub-field of the ring $\mathfrak{o}/\mathfrak{m}$. This being said, we set up the following definition:

DEFINITION 1. *A local ring is a Noetherian ring $\mathfrak{o}$ which satisfies the following conditions:* a) *the non-units in $\mathfrak{o}$ form an ideal $\mathfrak{m}$;* b) *the ring $\mathfrak{o}$ contains a sub-*

[7] Cf. Krull, l.c. note 3), p. 692.

field K with infinitely many elements[8] *such that* $\mathfrak{o}/\mathfrak{m}$ *is a finite K-module. The ring* $\mathfrak{o}/\mathfrak{m}$, *which is necessarily a field, is called the field of residues of* $\mathfrak{o}$. *Any sub-field K of* $\mathfrak{o}$ *which satisfies condition* 2) *is called a basic field of* $\mathfrak{o}$.

DEFINITION 2. *Let* $\mathfrak{o}$ *be a local ring, and let* $\mathfrak{m}$ *be the ideal of non-units in* $\mathfrak{o}$. *We shall say that* $\mathfrak{o}$ *is of dimension r if it is possible to find a set of r elements of* $\mathfrak{o}$ *which generates an ideal which is primary for* $\mathfrak{m}$, *but if it is impossible to find a set of less than r elements with the same property. Any set of r elements which generates an ideal which is primary for* $\mathfrak{m}$ *is then called a system of parameters in* $\mathfrak{o}$.[9]

We include in this definition the case $r = 0$ by making the convention that the ideal generated by the empty set is the zero-ideal.

Since $\mathfrak{m}$ is a maximal prime ideal, the condition for an ideal of being primary for $\mathfrak{m}$ is clearly equivalent to the condition of containing some power of $\mathfrak{m}$. Therefore, a local ring of dimension 0 is a local ring in which every non-unit is nilpotent.

PROPOSITION 1. *If a ring* $\mathfrak{o}'$ *is a homomorphic image of a local ring* $\mathfrak{o}$, *it is a local ring and we have* $\dim \mathfrak{o}' \leqq \dim \mathfrak{o}$. *In the case where* $\dim \mathfrak{o}' = \dim \mathfrak{o}$, *the images in* $\mathfrak{o}'$ *of the elements of a system of parameters in* $\mathfrak{o}$ *form a system of parameters in* $\mathfrak{o}'$.

By Proposition 1, §II, p. 693, we know that $\mathfrak{o}'$ is a semi-local ring; the only maximal prime ideal of $\mathfrak{o}'$ is clearly the image of the ideal of non units in $\mathfrak{o}$. The image of a basic field of $\mathfrak{o}$ is a basic field of $\mathfrak{o}'$, which proves that $\mathfrak{o}'$ is a local ring. If $x_1, \cdots, x_r$ are elements of $\mathfrak{o}$ which generate a primary ideal for the ideal $\mathfrak{m}$ of non units, their images in $\mathfrak{o}'$ generate an ideal which is primary for the ideal of non units of $\mathfrak{o}'$, which proves Proposition 1.

Since a local ring is also semi-local, we may use the notions of "complete", or of "completion", (and more generally all the topological notions), in connection with a local ring.

PROPOSITION 2. *The completion* $\bar{\mathfrak{o}}$ *of local ring* $\mathfrak{o}$ *of dimension r is a local ring of dimension r. Any system of parameters in* $\mathfrak{o}$ *is also a system of parameters in* $\bar{\mathfrak{o}}$.

If $\mathfrak{m}$ is the ideal of non units in $\mathfrak{o}$, we know that $\bar{\mathfrak{o}}$ is a semi-local ring in which every maximal prime ideal contains $\mathfrak{m}\bar{\mathfrak{o}}$; since $\mathfrak{o}$ is everywhere dense in $\bar{\mathfrak{o}}$, $\bar{\mathfrak{o}}/\mathfrak{m}\bar{\mathfrak{o}}$ is isomorphic with $\mathfrak{o}/\mathfrak{m}$, i.e. is a field. It follows that $\mathfrak{m}\bar{\mathfrak{o}}$ is prime and that a basic field of $\mathfrak{o}$ is also a basic field of $\bar{\mathfrak{o}}$:$\bar{\mathfrak{o}}$ is a local ring. The elements of a system of parameters in $\mathfrak{o}$ generate in $\bar{\mathfrak{o}}$ an ideal which is primary for $\mathfrak{m}\bar{\mathfrak{o}}$. Let conversely $\{\bar{x}_1, \cdots, \bar{x}_s\}$ be a system of parameters in $\bar{\mathfrak{o}}$, and assume that $\bar{\mathfrak{q}} = \sum_{i=1}^{s} \bar{\mathfrak{o}}\bar{x}_i$ contains $\mathfrak{m}^e\bar{\mathfrak{o}}$. We can find s elements $x_1, \cdots, x_s$ in $\mathfrak{o}$ such that $x_i \equiv \bar{x}_i \pmod{\mathfrak{m}^{e+1}\bar{\mathfrak{o}}}$ $(1 \leqq i \leqq s)$. If we set $\mathfrak{q} = \sum_{i=1}^{s} \bar{\mathfrak{o}}x_i$, we have $\bar{\mathfrak{q}} \subset \mathfrak{q} + \mathfrak{m}^{e+1}\bar{\mathfrak{o}}$, and, by induction, $\mathfrak{m}^e\bar{\mathfrak{o}} \subset \mathfrak{q} + \mathfrak{m}^{e+n}\bar{\mathfrak{o}}$ for every n. By Lemma 6, §II, p. 695, it follows that $\mathfrak{m}^e\bar{\mathfrak{o}} \subset \mathfrak{q}$, $\mathfrak{m}^e \subset \mathfrak{q} \cap \mathfrak{o} = \sum_{i=1}^{s} \mathfrak{o}x_i$, which proves that $\mathfrak{o}$ is of dimension $\leqq s$. Proposition 2 is proved.

PROPOSITION 3. *Let* $\mathfrak{o}$ *be a complete local ring and let* $\mathfrak{m}$ *be the ideal of non units*

[8] This restriction is not really necessary, but makes some of the arguments simpler.

[9] This definition of the dimension is equivalent to the one given by Krull, l.c. note 3), p. 692. (Cf. The appendix at the end of this paper.)

in $\mathfrak{o}$*; if there exists a basic field* K *of* $\mathfrak{o}$ *such that* $\mathfrak{o}/\mathfrak{m}$ *is separable over* K, K *is contained in a sub-field* K' *of* $\mathfrak{o}$ *which is a complete system of representatives for the residue classes of* $\mathfrak{o}$ *modulo* $\mathfrak{m}$.[10]

Let ζ be an element of $\mathfrak{o}$ whose residue class modulo $\mathfrak{m}$ generates $\mathfrak{o}/\mathfrak{m}$ over K. There exists an irreducible polynomial $F(Z)$ with coefficients in K such that $F(\zeta) \equiv 0 \pmod{\mathfrak{m}}$, $dF/dZ(\zeta) \not\equiv 0 \pmod{\mathfrak{m}}$. It follows that $dF/dZ(\zeta)$ has an inverse a in $\mathfrak{o}$. We shall construct by induction a sequence of elements $\zeta_k \in \mathfrak{o}$ such that $F(\zeta_k) \equiv 0 \pmod{\mathfrak{m}^k}$. We set $\zeta_1 = \zeta$; if ζ_k is already constructed, we set $\zeta_{k+1} = \zeta_k - aF(\zeta_k)$; it follows immediately from the Taylor expansion theorem that $F(\zeta_{k+1}) \equiv 0 \pmod{\mathfrak{m}^{2k}}$. Moreover our construction shows that the sequence (ζ_k) is convergent; if ζ_∞ is its limit, we have $F(\zeta_\infty) = 0$. Since ζ_∞ is algebraic over K, we have $K(\zeta_\infty) = K[\zeta_\infty] \subset \mathfrak{o}$, and $K(\zeta_\infty)$ is clearly a complete system of representatives for the classes modulo $\mathfrak{m}$.

PROPOSITION 4. *Let* $\mathfrak{o}$ *be a complete local ring, and let* $\{x_1, \cdots, x_r\}$ *be a system of parameters in* $\mathfrak{o}$. *If* K *is a basic field of* $\mathfrak{o}$, *the elements* $x_1, \cdots, x_r$ *are analytically independent over* K.

Proposition 3 will follow immediately from

LEMMA 1. *Let* $\mathfrak{o}$ *be a local ring; let* $\{x_1, \cdots, x_r\}$ *be a system of parameters in* $\mathfrak{o}$; *let* F *be a form of degree* d *in* r *letters with coefficients in* $\mathfrak{o}$; *if one at least of the coefficients of* F *does not belong to the ideal of non units* $\mathfrak{m}$ *of* $\mathfrak{o}$, *we have* $F(x_1, \cdots, x_r) \in \mathfrak{x}^{d+1}$, *where* $\mathfrak{x} = \sum_{i=1}^{r} \mathfrak{o}x_i$.[11]

Assume for a moment that $F(x_1, \cdots, x_r) \in \mathfrak{x}^{d+1}$; let F^* be the form with coefficients in $\mathfrak{o}/\mathfrak{m}$ which is deduced from F by reduction of the coefficients modulo $\mathfrak{m}$; since $\mathfrak{o}/\mathfrak{m}$ contains infinitely many elements, we can find $r-1$ elements $a_1^*, \cdots, a_{r-1}^*$ of this field such that $F^*(a_1^*, \cdots, a_{r-1}^*, 1) \neq 0$; we select for each i $(1 \leqq i \leqq r-1)$ a representative $a_i \in \mathfrak{o}$ of the class a_i^*, and we set $x_i' = x_i - a_i x_r$. Let $\mathfrak{x}'$ be the ideal $\sum_{i=1}^{r-1} \mathfrak{o}x_i'$. We have $F(x_1, \cdots, x_r) \equiv F(a_1, \cdots, a_{r-1}, 1)x_r^d \pmod{\mathfrak{x}'}$ and $F(a_1, \cdots, a_{r-1}, 1)$ is a unit; it follows that $x_r^d \in \mathfrak{x}' + \mathfrak{x}^{d+1}$; since $\mathfrak{x}^d \subset \mathfrak{x}' + \mathfrak{o}x_r^d$, we have $\mathfrak{x}^d \subset \mathfrak{x}' + \mathfrak{x}^{d+1}$, whence, by induction, $\mathfrak{x}^d \subset \mathfrak{x}' + \mathfrak{x}^{d+n}$ for every n; by Lemma 6, §II, p. 695, it follows that $\mathfrak{x}^d \subset \mathfrak{x}'$, whence $\mathfrak{m}^{dh} \subset \mathfrak{x}'$ for some h, which would imply $\dim \mathfrak{o} < r$.

PROPOSITION 5. *Let* $\mathfrak{o}$ *be a complete local ring, and let* $x_1, \cdots, x_m$ *be elements of* $\mathfrak{o}$ *which generate an ideal which is primary for the ideal* $\mathfrak{m}$ *of non-units in* $\mathfrak{o}$. *Then* $\mathfrak{o}$ *is a finite module over* $K[[x_1, \cdots, x_m]]$.

This follows immediately from Proposition 4, §II, p. 695.

PROPOSITION 6. *Let* $\mathfrak{o}$ *be a local ring, and let* $\mathfrak{a}$ *be an ideal in* $\mathfrak{o}$. *If* $\mathfrak{a}$ *contains an element which is not a zero divisor in* $\mathfrak{o}$, *we have* $\dim \mathfrak{o}/\mathfrak{a} < \dim \mathfrak{o}$.

Let $\bar{\mathfrak{o}}$ be the completion of $\mathfrak{o}$; we know that $\bar{\mathfrak{o}}/\mathfrak{a}\bar{\mathfrak{o}}$ is isomorphic with the completion of $\mathfrak{o}/\mathfrak{a}$ (Proposition 5, §II, p. 699). Moreover $\mathfrak{a}\bar{\mathfrak{o}}$ contains an element which is not a zero divisor in $\bar{\mathfrak{o}}$ (Proposition 6, §II, p. 699). It follows that it

[10] Mr. Cohen has proved that every complete local ring (in the sense of the definition of Krull) which contains a field contains a field which is a complete system of representatives for the residue classes modulo the maximal ideal.

[11] It can be proved that Lemma 1 still holds for any local ring in the sense of Krull.

is sufficient to prove our assertion in the case where $\mathfrak{o}$ is complete. Let K be a basic field of $\mathfrak{o}$, and let $\{x_1, \cdots, x_r\}$ be a system of parameters in $\mathfrak{o}$. If y is an element of $\mathfrak{a}$ which is not a zero divisor, y is integral over $K[[x_1, \cdots, x_r]]$ by Proposition 5: there exists a relationship of the form $y^m + b_1 y^{m-1} + \cdots + b_m = 0$, with $b_i \in K[[x_1, \cdots, x_r]]$ $(1 \leqq i \leqq m)$; since y is not a zero-divisor, we may assume that $b_m = F(x_1, \cdots, x_r) \neq 0$. We have $b_m \in \mathfrak{a}$, whence $F(x_1', \cdots, x_r') = 0$, if $x_1', \cdots, x_r'$ are the residue classes of $x_1, \cdots, x_r$ modulo $\mathfrak{a}$. By Proposition 4 above, it follows that $x_1', \cdots, x_r'$ do not form a system of parameters in $\mathfrak{o}/\mathfrak{a}$, whence $\dim \mathfrak{o}/\mathfrak{a} < r$ by Proposition 1.

PROPOSITION 7. *Let $x_1, \cdots, x_m$ be elements of a local ring $\mathfrak{o}$ of dimension r which generate an ideal which is primary for the ideal $\mathfrak{m}$ of non-units in $\mathfrak{o}$. Let K be a basic field of $\mathfrak{o}$; it is possible to find r linear combinations of $x_1, \cdots, x_m$ with coefficients in $\mathfrak{o}$ which form a system of parameters in $\mathfrak{o}$.*

We proceed by induction on r. The result is trivial for $r = 0$. Assume that $r > 0$ and that our result holds in local rings of dimension $r - 1$. Let $\mathfrak{x}$ be the ideal $\sum_{i=1}^{m} \mathfrak{o} x_i$, and let $\mathfrak{c}$ be the set of the elements $c \in \mathfrak{o}$ such that $c\mathfrak{x}^a = \{0\}$ holds for some a. It is clear that $\mathfrak{c}$ is an ideal; since it has a finite set of generators, we have $\mathfrak{c}\mathfrak{x}^{a_0} = \{0\}$ for some a_0; if $\mathfrak{m}^h \subset \mathfrak{x}$, we have $\mathfrak{m}^{a_0 h}\mathfrak{c} = \{0\}$, whence $\mathfrak{c}^{a_0 h+1} = \{0\}$, since $\mathfrak{c} \subset \mathfrak{m}$. We set $\mathfrak{o}' = \mathfrak{o}/\mathfrak{c}$, and we denote by $x_1', \cdots, x_m'$ the residue classes of $x_1, \cdots, x_m$ modulo $\mathfrak{c}$. If $c' \in \mathfrak{o}'$, the conditions $c'x_i' = 0$ $(1 \leqq i \leqq m)$ imply $c' = 0$. Let $\mathfrak{n}_1, \cdots, \mathfrak{n}_g$ be the prime divisors of the zero ideal in $\mathfrak{o}'$: none of these ideals can contain all the elements x_i' at the same time. Let $\mathfrak{M}$ be the vector space over K spanned by $x_1', \cdots, x_m'$; if $\mathfrak{M}_i = \mathfrak{M} \cap \mathfrak{n}_i$, we have $\mathfrak{M}_i \neq \mathfrak{M}$ $(1 \leqq i \leqq g)$; since K contains infinitely many elements, we can find a linear combination y_1' of $x_1', \cdots, x_m'$ with coefficients in K which does not belong to any one of the sets $\mathfrak{M}_i$: y_1' is not a zero-divisor in $\mathfrak{o}'$. We denote by r' the dimension of $\mathfrak{o}'$, whence $r' \leqq r$. Let $\mathfrak{o}''$ be the ring $\mathfrak{o}'/\mathfrak{o}'y_1'$: we have $\dim \mathfrak{o}'' < r'$ by Proposition 5. If $y_2', \cdots, y_s'$ are elements of $\mathfrak{o}'$ whose residue classes modulo $\mathfrak{o}'y_1'$ form a system of parameters in $\mathfrak{o}''$, it is clear that the ideal $\sum_{i=1}^{s} \mathfrak{o}'y_i'$ is primary for the ideal of non-units in $\mathfrak{o}'$, whence $s + 1 \leqq r'$. It follows that $s = r' - 1$. In virtue of our induction assumption, we may take for $y_2', \cdots, y_m'$, linear combinations of $x_1', \cdots, x_m'$ with coefficients in K; the elements $y_1', \cdots, y_{r'}'$ form a system of parameters in $\mathfrak{o}'$ which is composed of linear combinations of $x_1', \cdots, x_m'$ with coefficients in K. Let $y_1, \cdots, y_{r'}$ be linear combinations of $x_1, \cdots, x_m$ with coefficients in K belonging respectively to the residue classes $y_1', \cdots, y_{r'}'$ modulo $\mathfrak{c}$, and let $\mathfrak{m}_0$ be any prime ideal containing $y_1, \cdots, y_{r'}$. The ideal of non-units in $\mathfrak{o}'$ being $\mathfrak{m}/\mathfrak{c}$, we see that $\mathfrak{m}^k \subset \mathfrak{m}_0 + \mathfrak{c}$ for some k. It follows that $\mathfrak{m}^{k(a_0 h+1)} \subset \mathfrak{m}_0$, whence $\mathfrak{m}_0 = \mathfrak{m}$, which proves that $r' = r$ and that $y_1, \cdots, y_r$ form a system of parameters in $\mathfrak{o}$.

COROLLARY. *Let $\mathfrak{o}$ be a complete local ring, and let K be a basic field of $\mathfrak{o}$. If $x_1, \cdots, x_r$ are elements of $\mathfrak{o}$ which are analytically independent over K and such that $\mathfrak{o}$ is a finite module over $K[[x_1, \cdots, x_r]]$, these elements form a system of parameters in $\mathfrak{o}$.*

Let $\mathfrak{m}$ be any prime ideal in $\mathfrak{o}$ containing $x_1, \cdots, x_r$; the ring $\mathfrak{o}/\mathfrak{m}$ is a finite

module over $K[[x_1, \cdots, x_r]]/K[[x_1, \cdots, x_r]] \cap \mathfrak{m} = K$; since $\mathfrak{o}/\mathfrak{m}$ is a ring without zero divisors $\neq 0$, $\mathfrak{o}/\mathfrak{m}$ is a field, which proves that $\mathfrak{m}$ is the ideal of non-units in $\mathfrak{o}$. By Proposition 7, we can form s linear combinations $y_1, \cdots, y_s$ of $x_1, \cdots, x_r$ with coefficients in K which constitute a system of parameters in $\mathfrak{o}$; since $K[[x_1, \cdots, x_r]]$ is a finite module over $K[[y_1, \cdots, y_s]]$, we must obviously have $s = r$, which proves the corollary.

PROPOSITION 8. *Let $\mathfrak{o}'$ be a ring which contains a complete local ring $\mathfrak{o}$ as a sub-ring and is a finite $\mathfrak{o}$-module. If $\mathfrak{o}'$ has no zero divisor $\neq 0$, $\mathfrak{o}'$ is a complete local ring. If $\mathfrak{m}$ is the ideal of non-units in $\mathfrak{o}$, $\mathfrak{m}\mathfrak{o}'$ is primary for the ideal of non-units in $\mathfrak{o}'$.*

By Proposition 3, §II, p. 694 we know that $\mathfrak{o}'$ is a complete semi-local ring and that $\mathfrak{m}\mathfrak{o}'$ contains a power of the product of all maximal prime ideals $\mathfrak{m}'$ of $\mathfrak{o}'$. By Proposition 2, §II, p. 693, we see that, $\mathfrak{o}'$ having no zero-divisor $\neq 0$, there can exist only one prime ideal $\mathfrak{m}'$. Proposition 8 then follows immediately.

DEFINITION 3. *A local ring $\mathfrak{o}$ is said to be regular*[12] *if the ideal $\mathfrak{m}$ of non-units in $\mathfrak{o}$ can be generated by a system of parameters; such a system of parameters is then said to be regular.*

From now on, we shall denote by $\mathfrak{o}$ a regular local ring and by $\{x_1, \cdots, x_r\}$ a regular system of parameters in $\mathfrak{o}$.

It is clear that the completion of $\mathfrak{o}$ is again regular; if $\{u_1, \cdots, u_d\}$ is a set of elements of $\mathfrak{o}$ whose residue classes modulo $\mathfrak{m}$ form a linear base of $\mathfrak{o}/\mathfrak{m}$ with respect to a basic field of $\mathfrak{o}$, we have $\bar{\mathfrak{o}} = \sum_{i=1}^{d} K[[x_1, \cdots, x_r]]u_i$.[13] Every element $a \in \mathfrak{m}^k$ may be written in the form $F(x_1, \cdots, x_r)$, where F is a form of degree k with coefficients in $\mathfrak{o}$; by Lemma 1, the form F^* deduced from F by reduction of the coefficients modulo $\mathfrak{m}$ is uniquely determined when a is given; this form is null if and only if $a \in \mathfrak{m}^{k+1}$. It follows that, if $a \in \mathfrak{m}^k$, $a \notin \mathfrak{m}^{k+1}$, $b \in \mathfrak{m}^l$, $b \notin \mathfrak{m}^{l+1}$, we have $ab \notin \mathfrak{m}^{k+l+1}$. In particular, a regular local ring has no zero divisor $\neq 0$.

Let now $\mathfrak{a}$ be the ideal generated in $\mathfrak{o}$ by $x_1, \cdots, x_{r-s}$, where $s < r$. The residue classes $x_{r-s+1}^*, \cdots, x_r^*$ of $x_{r-s+1}, \cdots, x_r$ modulo $\mathfrak{a}$ generate the ideal of non units in $\mathfrak{o}/\mathfrak{a}$; conversely, if $y_1, \cdots, y_t$ are elements of $\mathfrak{o}$ whose residue classes modulo $\mathfrak{a}$ form a system of parameters in $\mathfrak{o}/\mathfrak{a}$, the ideal generated by $x_1, \cdots, x_{r-s}, y_1, \cdots, y_t$ is primary for $\mathfrak{m}$, whence $\dim \mathfrak{o}/\mathfrak{a} = t \geqq r - s$. It follows that $\mathfrak{o}/\mathfrak{a}$ is regular and that $x_{r-s+1}^*, \cdots, x_r^*$ form a regular system of parameters in $\mathfrak{o}/\mathfrak{a}$. Since $\mathfrak{o}/\mathfrak{a}$ has no zero divisor $\neq 0$, $\mathfrak{a}$ is prime.

Conversely, let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$ such that $\mathfrak{o}/\mathfrak{p}$ is regular, and let $y_1, \cdots, y_s$ be elements of $\mathfrak{o}$ whose residue classes modulo $\mathfrak{p}$ form a regular system of parameters in $\mathfrak{o}/\mathfrak{p}$. We can express y_i in the form $\sum_{j=1}^{n} a_{ij}x_j (a_{ij} \in \mathfrak{o})$; let a_{ij}^* be the residue class of a_{ij} modulo $\mathfrak{p}$. Since the residue classes $x_1^*, \cdots, x_r^*$ of $x_1, \cdots, x_r$ clearly form a system of generators of $\mathfrak{o}/\mathfrak{p}$, we have also $x_j^* =$

[12] This notion corresponds to the notion of "p. Stellenring" of Krull, l.c. note 3), p. 692. We use the word "regular" because these rings occur in the theory of the regular points of an algebraic variety.

[13] Cf. the proof of Proposition 4, §II, p. 695.

$\sum_{k=1}^{s} b_{jk}^{*} y_k^{*}$; let A^*, B^* be the rectangular matrices (a_{ij}^{*}), (b_{jk}^{*}), and let $C^* = (c_{ik}^{*})$ be the matrix A^*B^*. We have $y_i^{*} = \sum_{k=1}^{s} c_{ik}^{*} y_k^{*}$; by Lemma 1, we have $c_{ik}^{*} \equiv \delta_{ik} \pmod{\mathfrak{m}/\mathfrak{p}}$. It follows that the determinant of C^* is a unit, which proves that one at least of the determinants of order s extracted from A^* is a unit; we may assume without loss of generality that this happens for the determinant of the s last columns. We may then express $x_{r-s+1}, \cdots, x_r$ as linear combinations of $y_1, \cdots, y_s, x_1, \cdots, x_{r-s}$ with coefficients in $\mathfrak{o}$, whence $\mathfrak{m} = \sum_{i=1}^{s} \mathfrak{o} y_i + \sum_{h=1}^{r-s} \mathfrak{o} x_h$. Let b_{jk} be a representative in $\mathfrak{o}$ of the residue class b_{jk}^{*} modulo $\mathfrak{p}$; we set $x_h' = x_h - \sum_{k=1}^{s} b_{h,k} y_k$. It is clear that $\{y_1, \cdots, y_s, x_1', \cdots, x_{r-s}'\}$ is a regular system of parameters in $\mathfrak{o}$ and that $x_h' \in \mathfrak{p}$ $(1 \leqq h \leqq r - s)$. The ideal $\mathfrak{p}' = \sum_{h=1}^{r-s} \mathfrak{o} x_h'$ is contained in $\mathfrak{p}$; we know already that it is a prime ideal and that $\dim \mathfrak{o}/\mathfrak{p}' = s = \dim \mathfrak{o}/\mathfrak{p}$. By Proposition 6, it follows that $\mathfrak{p} = \mathfrak{p}'$. We have therefore proved the following results:

PROPOSITION 9. *Let $\mathfrak{o}$ be a regular local ring. Then $\mathfrak{o}$ does not contain any zero divisor $\neq 0$. If $\{x_1, \cdots, x_r\}$ is a regular system of parameters in $\mathfrak{o}$, and if $s < r$, the ideal generated in $\mathfrak{o}$ by $x_1, \cdots, x_{r-s}$ is prime and the factor ring of $\mathfrak{o}$ by this ideal is regular of dimension s. Conversely, let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$ such that $\mathfrak{o}/\mathfrak{p}$ is regular, and let $y_1, \cdots, y_s$ be elements of $\mathfrak{o}$ whose residue classes modulo $\mathfrak{p}$ form a regular system of parameters in $\mathfrak{o}/\mathfrak{p}$. Then we can find $r - s$ elements in $\mathfrak{o}$ which form a set of generators of $\mathfrak{p}$ and which, together with $y_1, \cdots, y_s$, form a regular system of parameters in $\mathfrak{o}$.*

§IV. Definition of the Multiplicity

LEMMA 1. *Let $\mathfrak{S}$ be a hypercomplex system over a field K, and let $\{0\} = \bigcap_{i=1}^{g} \mathfrak{q}_i$ be an irredundant representation of the zero-ideal in $\mathfrak{S}$ as an intersection of primary ideals. If $\mathfrak{u}_i$ is the prime divisor of $\mathfrak{q}_i$, we have $[\mathfrak{S}:K] = \sum_{i=1}^{g} [\mathfrak{S}/\mathfrak{u}_i:K] \cdot l_i$, where l_i is the length of $\mathfrak{q}_i$.*

It is well known that $\mathfrak{S}$ is the direct sum of g ideals $\mathfrak{S}_i = \mathfrak{S}\epsilon_i$, where each ϵ_i is an idempotent and $\mathfrak{S}_i$ is isomorphic (as a ring) with $\mathfrak{S}/\mathfrak{q}_i$; $\mathfrak{S}_i$ may be considered as a "primary" hypercomplex system over $K\epsilon_i$, in which the only prime ideal is $\mathfrak{u}_i/\mathfrak{q}_i$. It follows that it will be sufficient to prove the lemma in the case where $g = 1$. We then set $\mathfrak{u}_1 = \mathfrak{u}$, $l_1 = l$, and we have to prove that $[\mathfrak{S}:K] = [\mathfrak{S}/\mathfrak{u}:R] \cdot l$. Let us form a chain of $l + 1$ ideals $\mathfrak{v}_0, \mathfrak{v}_1, \cdots, \mathfrak{v}_l$, beginning with $\mathfrak{v}_0 = \{0\}$, ending with $\mathfrak{v}_l = \mathfrak{S}$, such that $\mathfrak{v}_{j-1} \subset \mathfrak{v}_j$, $\mathfrak{v}_{j-1} \neq \mathfrak{v}_j$ and that no further ideal can be inserted between $\mathfrak{v}_{j-1}$ and $\mathfrak{v}_j$ $(1 \leqq j \leqq l)$. Let v_j $(1 \leqq j \leqq l)$ be an element of $\mathfrak{v}_j$ not contained in $\mathfrak{v}_{j-1}$; we have clearly $\mathfrak{v}_j = \mathfrak{S}v_1 + \cdots + \mathfrak{S}v_j$. We shall prove that $\mathfrak{u}\mathfrak{v}_j \subset \mathfrak{v}_{j-1}$; in fact, the ideal $\mathfrak{v}_{j-1} + \mathfrak{u}\mathfrak{v}_j$ is either equal to $\mathfrak{v}_{j-1}$ (in which case our assertion is proved) or to $\mathfrak{v}_j$; in the latter case, we would have $v_j(1 + u) \in \mathfrak{v}_{j-1}$ for some $u \in \mathfrak{u}$, which is impossible because $1 + u$ is clearly a unit in $\mathfrak{S}$. It follows that $\mathfrak{v}_j\mathfrak{v}/\mathfrak{v}_{j-1}$ may be considered as an $(\mathfrak{S}/\mathfrak{u})$-module. Every $(\mathfrak{S}/\mathfrak{u})$-sub-module of $\mathfrak{v}_j/\mathfrak{v}_{j-1}$ is of the form $\mathfrak{v}/\mathfrak{v}_{j-1}$ where $\mathfrak{v}$ is an ideal in $\mathfrak{S}$ such that $\mathfrak{v}_{j-1} \subset \mathfrak{v} \subset \mathfrak{v}_j$; therefore $\mathfrak{v}/\mathfrak{v}_{j-1}$ is an $(\mathfrak{S}/\mathfrak{u})$-module of dimension 1, whence $[\mathfrak{v}_j/\mathfrak{v}_{j-1}:K] = [\mathfrak{S}/\mathfrak{u}:K]$. Since $[\mathfrak{S}:K] = \sum_{j=1}^{g} [\mathfrak{v}_j/\mathfrak{v}_{j-1}:K]$, Lemma 1 is proved.

LEMMA 2. *Let $\mathfrak{r}$ be a local ring of dimension 1 in which the ideal of non units*

is principal, and let x be a generator of this ideal. Let $\mathfrak{o}$ be a ring containing $\mathfrak{r}$ and which is a finite $\mathfrak{r}$-module. Assume that no element $\neq 0$ if $\mathfrak{r}$ is a zero divisor in $\mathfrak{o}$. Let $x\mathfrak{o} = \bigcap_{i=1}^{g} \mathfrak{q}_i$ be an irredundant representation of $x\mathfrak{o}$ as an intersection of primary ideals in $\mathfrak{o}$. We have $[\mathfrak{o}:\mathfrak{r}] = \sum_{i=1}^{g} [\mathfrak{o}/\mathfrak{u}_i:\mathfrak{r}/x\mathfrak{r}]\cdot l_i$, where $\mathfrak{u}_i$ is the associated prime ideal of $\mathfrak{q}_i$ and where l_i is the length of $\mathfrak{q}_i$.[14]

It is clear that every ideal in $\mathfrak{r}$ is principal and is generated by some power of x. It follows that $\mathfrak{o}$ has a linear base $\{\omega_1, \cdots, \omega_d\}$ over $\mathfrak{r}$ (the elements $\omega_1, \cdots, \omega_d$ being linearly independent over $\mathfrak{r}$). Since $\mathfrak{u}_i \neq \mathfrak{o}$ and since $x\mathfrak{r}$ is a maximal prime ideal in $\mathfrak{r}$, we have $\mathfrak{u}_i \cap \mathfrak{r} = x\mathfrak{r}$ $(1 \leqq i \leqq g)$. The ring $\mathfrak{o}/x\mathfrak{o}$ is a finite module over the field $\mathfrak{r}/x\mathfrak{r}$; this means that $\mathfrak{o}/x\mathfrak{o}$ is a hypercomplex system over $\mathfrak{r}/x\mathfrak{r}$. It is clear that the residue classes of $\omega_1, \cdots, \omega_d$ modulo $x\mathfrak{o}$ form a linear base of $\mathfrak{o}/x\mathfrak{o}$ over $\mathfrak{r}/x\mathfrak{r}$, whence $[\mathfrak{o}/x\mathfrak{o}:\mathfrak{r}/x\mathfrak{r}] = [\mathfrak{o}:\mathfrak{r}]$.

It is clear that $\{0\} = \bigcap_{i=1}^{g} \mathfrak{q}_i/x\mathfrak{o}$ is an irredundant representation of the zero ideal in $\mathfrak{o}/x\mathfrak{o}$ as an intersection of primary ideals and that the associated prime ideal of $\mathfrak{q}_i/x\mathfrak{o}$ is $\mathfrak{u}_i/x\mathfrak{o}$. The length of $\mathfrak{q}_i/x\mathfrak{o}$ is equal to l_i, because there exists a one-to-one inclusion preserving correspondence between the ideals containing $\mathfrak{q}_i$ in $\mathfrak{o}$ and the ideals containing $\mathfrak{q}_i/x\mathfrak{o}$ in $\mathfrak{o}/x\mathfrak{o}$. The field $(\mathfrak{o}/x\mathfrak{o})/(\mathfrak{u}_i/x\mathfrak{o})$ is isomorphic to $\mathfrak{o}/\mathfrak{u}_i$; therefore Lemma 2 follows immediately from Lemma 1.

THEOREM 2. *Let $\mathfrak{o}$ be a complete local ring which does not contain any zero divisor $\neq 0$, and let $\{x_1, \cdots, x_r\}$ be a system of parameters in $\mathfrak{o}$. There exists an integer $e = e(\mathfrak{o}; x_1, \cdots, x_r)$ which has the following property; if K is any basic field of $\mathfrak{o}$, we have $[\mathfrak{o}:K[[x_1, \cdots, x_r]]] = e$. $[\mathfrak{o}/\mathfrak{m}:K]$, where $\mathfrak{m}$ is the ideal of non units in $\mathfrak{o}$.*

The proof proceeds by induction on the dimension r of $\mathfrak{o}$. If $r = 1$, the result follows immediately from Lemma 2. Assume that $r > 1$ and that the result holds for the dimension $r - 1$. If K is a basic field of $\mathfrak{o}$, an element of the form $x_r^{-1}F(x_1, \cdots, x_{r-1})$ (where F is a power series with coefficients in K) is not integral over $K[[x_1, \cdots, x_r]]$ (because this ring is integrally closed) and therefore does not belong to $\mathfrak{o}$; it follows that $x_r\mathfrak{o} \cap K[[x_1, \cdots, x_r]] = x_rK[[x_1, \cdots, x_r]]$. The ideal $x_r\mathfrak{o}$ contains some power product of prime ideals in $\mathfrak{o}$; if every one of these prime ideals were to contain an element $\neq 0$ of $K[[x_1, \cdots, x_{r-1}]]$ the same would hold for $x_r\mathfrak{o}$ (because $K[[x_1, \cdots, x_{r-1}]]$ has no zero-divisor $\neq 0$); this is not being the case, there exists at least one prime ideal $\mathfrak{u}$ in $\mathfrak{o}$ containing x_r and such that $\mathfrak{u} \cap K[[x_1, \cdots, x_{r-1}]] = \{0\}$. If $\mathfrak{u}$ is any such ideal, the residue classes $x_1', \cdots, x_{r-1}'$ of $x_1, \cdots, x_{r-1}$ modulo $\mathfrak{u}$ are analytically independent over K and $\mathfrak{o}/\mathfrak{u}$ is a finite module over the ring $K[[x_1', \cdots, x_{r-1}']]$, whence $\dim \mathfrak{o}/\mathfrak{u} = r - 1$, by the corollary to Proposition 7, §III, p. 703. Conversely, if $\mathfrak{u}$ is a prime ideal containing x_r and such that $\dim \mathfrak{o}/\mathfrak{u} = r - 1$, the residue classes of $x_1, \cdots, x_{r-1}$ modulo $\mathfrak{u}$ obviously form a system of parameters in $\mathfrak{o}/\mathfrak{u}$, whence $\mathfrak{u} \cap K[[x_1, \cdots, x_{r-1}]] = \{0\}$. We see that the prime ideals $\mathfrak{u}$ containing x_r and which satisfy the latter condition are characterized independently of the basic field K; it follows immediately from Proposition 6, §III, p. 702 that any

[14] We denote by $[\mathfrak{o}:\mathfrak{r}]$ the maximum number of elements of $\mathfrak{o}$ linearly independent over $\mathfrak{r}$. This is also the dimension of the ring of quotients of $\mathfrak{o}$, considered as a vector space over the field of quotients of $\mathfrak{r}$.

one of these ideals is a minimal prime divisor of $x_r\mathfrak{o}$, which proves that there are only a finite number of them, say $\mathfrak{u}_1, \cdots, \mathfrak{u}_g$; moreover there corresponds to every $\mathfrak{u}_j$ a uniquely defined primary component $\mathfrak{v}_j$ of $x_r\mathfrak{o}$; we denote by l_j the length of $\mathfrak{v}_j$. Since $\mathfrak{o}/\mathfrak{u}_j$ is of dimension $r-1$, our induction assumption guarantees the existence of a number $e_j = e(\mathfrak{o}/\mathfrak{u}_j; x_{1,j}, \cdots, x_{r-1,j})$ (where $x_{i,j}$ is the residue class of x_i modulo $\mathfrak{u}_j$) such that $[\mathfrak{o}/\mathfrak{u}_j : K[[x_{1,j}, \cdots, x_{r-1,j}]]] = e_j \cdot [(\mathfrak{o}/\mathfrak{u}_j)/(\mathfrak{m}/\mathfrak{u}_j) : K]$. We shall see that the number $e = \sum_j e_j l_j$ has the required property.

We denote by S the set of the elements $\neq 0$ in $K[[x_1, \cdots, x_{r-1}]]$; S is multiplicatively closed and no element of S is a zero divisor in $\mathfrak{o}$; it follows that we can construct the rings of quotients $\mathfrak{o}_S$ of $\mathfrak{o}$ and $\mathfrak{r}$ of $K[[x_1, \cdots, x_r]]$ with respect to S. The ideals $\mathfrak{u}_j$ are all the prime divisors of $x\mathfrak{o}$ which do not meet S; by Lemma 3, §I, p. 691, $x_r\mathfrak{o}_S = \bigcap_{j=1}^{g} \mathfrak{v}_j\mathfrak{o}_S$ is an irredundant representation of $x_r\mathfrak{o}$ as intersection of primary ideals in $\mathfrak{o}_S$, and $\mathfrak{v}_j\mathfrak{o}_S$ is of length l_j. The ring $\mathfrak{r}$ is clearly a local ring of dimension 1 in which the ideal of non units is generated by x_r; since $\mathfrak{o}$ is finite over $K[[x_1, \cdots, x_r]]$, $\mathfrak{o}_S$ is finite over $\mathfrak{r}$. The field $\mathfrak{o}_S/\mathfrak{u}_j\mathfrak{o}_S$ is the field of quotients of $\mathfrak{o}/\mathfrak{u}_j$, whence $[\mathfrak{o}_S/\mathfrak{u}_j\mathfrak{o}_S : \mathfrak{r}/\mathfrak{u}_j\mathfrak{o}_S \cap \mathfrak{r}] = e_j \cdot [(\mathfrak{o}/\mathfrak{u}_j)/(\mathfrak{m}/\mathfrak{u}_j) : K] = e_j[\mathfrak{o}/\mathfrak{m} : K]$. By Lemma 2, we have $[\mathfrak{o}_S : \mathfrak{r}] = e[\mathfrak{o}/\mathfrak{m} : K]$, which proves Theorem 2 for the dimension r, since $[\mathfrak{o}_S : \mathfrak{r}] = [\mathfrak{o} : K[[x_1, \cdots, x_r]]]$.

DEFINITION 1. *If $\mathfrak{o}$ is a complete local ring without any zero divisor $\neq 0$, and if $\{x_1, \cdots, x_r\}$ is a system of parameters in $\mathfrak{o}$, the number $e(\mathfrak{o}; x_1, \cdots, x_r)$ whose existence is asserted in Theorem 2 is called the multiplicity of $\mathfrak{o}$ with respect to the system $\{x_1, \cdots, x_r\}$.*

It is clear that this multiplicity is an invariant of the ring $K[[x_1, \cdots, x_r]]$, where K is any basic field of $\mathfrak{o}$. We shall prove that it is also an invariant of the ideal $\sum_{i=1}^{r} \mathfrak{o}x_i$.

PROPOSITION 1. *Let $\mathfrak{o}$ be a complete local ring which does not contain any zero divisor $\neq 0$. If $\{x_1, \cdots, x_r\}$ and $\{x_1', \cdots, x_r'\}$ are two systems of parameters in $\mathfrak{o}$, the equality $\sum_1^r \mathfrak{o}x_i = \sum_1^r \mathfrak{o}x_i'$ implies $e(\mathfrak{o}; x_1, \cdots, x_r) = e(\mathfrak{o}; x_1', \cdots, x_r')$.*

We set $x_i' = \sum_{j=1}^{r} a_{ij}x_j$ $(a_{ij} \in \mathfrak{o})$, and we denote by $\mathfrak{m}$ the ideal of non-units in $\mathfrak{o}$. One at least of the elements a_{ij} does not belong to $\mathfrak{m}$, because otherwise we would have $\sum_1^r \mathfrak{o}x_i' \subset \mathfrak{m}(\sum_1^r \mathfrak{o}x_i)$, which is clearly impossible if $\sum_1^r \mathfrak{o}x_i = \sum_1^r \mathfrak{o}x_i'$. We may assume without loss of generality that $a_{rr} \notin \mathfrak{m}$; it follows that a_{rr} is a unit, whence $x_r \in \sum_{i=1}^{r-1} x_i\mathfrak{o} + x_r'\mathfrak{o}$ and $\sum_1^r \mathfrak{o}x_i = \sum_{i=1}^{r-1} \mathfrak{o}x_i + \mathfrak{o}x_r'$. It follows that, if $r > 1$, it is sufficient to prove Proposition 1 under the supplementary assumption that $\{x_1, \cdots, x_r\}$ and $\{x_1', \cdots, x_r'\}$ have an element in common.

This being said, we prove Proposition 1 by induction on r. If $r = 1$, Lemma 2 shows that $e(\mathfrak{o}; x_1)$ is the length of the primary ideal $\mathfrak{o}x_1$, which proves Proposition 1 in this case. Assume now that $r > 1$ and that $x_r' = x_r$. Making use of the notations of the proof of Theorem 2, we see that it is sufficient to prove that $e(\mathfrak{o}/\mathfrak{u}_j; x_{1,j}, \cdots, x_{r-1,j}) = e(\mathfrak{o}/\mathfrak{u}_j; x_{1,j}', \cdots, x_{r-1,j}')$, where $x_{i,j}, x_{i,j}'$ are the residue classes of x_i, x_i' modulo $\mathfrak{u}_j$. But we have $\sum_{i=1}^{r-1} (\mathfrak{o}/\mathfrak{u}_j)x_{i,j} = (\mathfrak{u}_j + \sum_{i=1}^{r} \mathfrak{o}x_i)/\mathfrak{u}_j = (\mathfrak{u}_j + \sum_{i=1}^{r} \mathfrak{o}x_i')/\mathfrak{u}_j = \sum_{i=1}^{r-1} (\mathfrak{o}/\mathfrak{u}_j)x_{i,j}'$, which proves the formula in question, since $\dim \mathfrak{o}/\mathfrak{u}_j = r - 1$.

APPENDIX

On the dimension of local rings

We shall prove that the definition of the dimension of a local ring which is given in the text (Definition 2, §III, p. 701) is equivalent to the definition given by Krull (loc. cit., note 3, p. 692). In this appendix, we shall take the notion of local ring in the sense of Krull (i.e., we drop from our definition the requirement of existence of a basic field).

Let $\mathfrak{o}$ be a local ring; we denote by r the dimension of $\mathfrak{o}$ in the sense of our definition and by r' its dimension in the sense of Krull's definition. The inequality $r' \leqq r$ follows immediately from Theorem 7* in Krull's paper; we have therefore to prove that $r \leqq r'$; it will be sufficient to prove that there exists a chain of $r+1$ prime ideals $\mathfrak{p}_0, \mathfrak{p}_1, \cdots, \mathfrak{p}_r$ in $\mathfrak{o}$ such that $\mathfrak{p}_{i-1}$ is properly contained in $\mathfrak{p}_i$ $(1 \leqq i \leqq r)$. There is nothing to prove if $r = 0$. Assume that $r > 0$ and that our assertion holds for local rings of dimension $r-1$. We denote by $\mathfrak{p}$ the ideal of non units in $\mathfrak{o}$ and by $\mathfrak{n}_1, \cdots, \mathfrak{n}_g$ the distinct minimal prime divisors of the zero ideal in $\mathfrak{o}$. We have $\mathfrak{n}_i \neq \mathfrak{p}$ $(1 \leqq i \leqq g)$ because $r \neq 0$; it follows that none of the ideals $\mathfrak{n}_i$ can contain simultaneously all elements of a system of parameters $\{x_1, \cdots, x_r\}$ in $\mathfrak{o}$. Assume for instance that $x_1 \notin \mathfrak{n}_1$. It may happen that x_1 belongs to some $\mathfrak{n}_k$ with $k > 1$; if this is the case, we select an index $i(k)$ such that $x_{i(k)} \notin \mathfrak{n}_k$ and an element a_k which belongs to $\prod_{j \neq k} \mathfrak{n}_j$ but not to $\mathfrak{n}_k$ (this is possible because $\mathfrak{n}_k$ does not contain $\mathfrak{n}_j$ for $j \neq k$); we set $x_1' = x_1 + \sum_k a_k x_{i(k)}$, the sum being extended to all indices k such that $x_1 \in \mathfrak{n}_k$. The element x_1' does not belong to any of the ideals $\mathfrak{n}_i$ $(1 \leqq i \leqq g)$ and the ideal $\mathfrak{o}x_1' + \sum_{i=2}^{r} \mathfrak{o}x_i$ is equal to $\sum_1^r \mathfrak{o}x_i$, which proves that $x_1', x_2, \cdots, x_r$ form a system of parameters in $\mathfrak{o}$. Let $\mathfrak{o}^*$ be the local ring $\mathfrak{o}/\mathfrak{o}x_1'$; if $y_1, \cdots, y_s$ are elements of $\mathfrak{o}$ whose residue classes modulo $\mathfrak{o}x_1'$ form a system of parameters in $\mathfrak{o}^*$, there exists an exponent h such that $\mathfrak{p}^h + \mathfrak{o}x_1' \subset \mathfrak{o}x_1' + \sum_{i=s}^{s} \mathfrak{o}y_j$, and therefore the ideal $\mathfrak{o}x_1' + \sum_{j=1}^{s} \mathfrak{o}y_j$ is primary for $\mathfrak{p}$, whence $s \geqq r-1$. Since the residue classes of $x_2, \cdots, x_r$ clearly generate an ideal in $\mathfrak{o}^*$ which is primary for $\mathfrak{p} + \mathfrak{o}x_1'/\mathfrak{o}x_1'$, we conclude that $\mathfrak{o}^*$ is of dimension $r-1$. It follows from our inductive assumption (applied to $\mathfrak{o}^*$) that there exists a chain of r distinct prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ in $\mathfrak{o}$, containing $\mathfrak{o}x_1'$ and such that $\mathfrak{p}_{i-1} \subset \mathfrak{p}_i$ $(2 \leqq i \leqq r)$. Since $\mathfrak{p}_1$ contains x_1', it cannot be a minimal prime divisor of the zero ideal. Since $\mathfrak{p}_1$ is prime and contains 0, it contains some minimal prime divisor of the zero ideal, say $\mathfrak{p}_0$, and our assertion is proved for local rings of dimension r.

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ON SOME ALGEBRAICAL PROPERTIES OF OPERATOR RINGS

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§1. The notations to be used in this paper agree with those of the papers quoted below, especially [2]. The results which we obtain will be used in [5], but they seem to have a certain interest of their own as well.

BIBLIOGRAPHY

[1] J. VON NEUMANN: "*Zur Algebra der Funktionaloperatoren . . .*," Math. Annalen, vol. 102, pp. 370–427 (1929).

[2] J. VON NEUMANN: "*Über adjungierte Funktionaloperatoren,*" Annals of Math., vol. 33, pp. 294–310 (1932).

[3] F. J. MURRAY: "*Linear transformations between Hilbert spaces,*" Trans. Amer. Math. Soc., vol. 37, 301–338 (1935).

[4] F. J. MURRAY AND J. VON NEUMANN: "*On rings of operators,*" Annals of Math., vol. 37, pp. 116–229 (1936).

[5] F. J. MURRAY AND J. VON NEUMANN: "*On rings of operators IV,*" Annals of Math., vol. 44, (following immediately).

§2. We consider an operator ring $\mathbf{M}$ in a Hilbert space $\mathfrak{H}$ which contains 1. We say that a notion defined in $\mathbf{M}$ is *purely algebraical* if it can be expressed in terms of the entity 1 (the unit operator) and the operations αA (α any complex number), A^*, $A + B$, AB alone and referring only to operators belonging to $\mathbf{M}$. Thus if a mapping of $\mathbf{M}$ onto another ring is isomorphic, that is if the notions of 1, αA, A^*, $A + B$, AB are invariant, then every "purely algebraic" notion is invariant.

Notions which refer explicitly to elements of $\mathfrak{H}$ will not in general be purely algebraical.

Now the notions which follow do refer to elements of $\mathfrak{H}$.

(α) Definiteness of an $A \in \mathbf{M}$.

(β) The numerical value of the bound $|||\, A \,|||$ of an $A \in \mathbf{M}$.

(γ) The fact that $\lim_{n\to\infty}$ strong $A_n = A$ when $A, A_1, A_2, \cdots \in \mathbf{M}$.

(δ) The fact that $\lim_{n\to\infty}$ weak $A_n = A$ when $A, A_1, A_2, \cdots \in \mathbf{M}$.

(For the *strong* and *weak* notions of convergence, used above, and for the corresponding topologies, to be referred to below, cf. [1], pp. 381–388.) The object of this paper is to show that the notions (α)–(δ) are nevertheless purely algebraical.

We shall not prove the same thing concerning the strong and weak topology (for operators). It is probably not generally true. For an important special case where it is true, cf. [5], Theorem I.

(α) and (β) are easy to dispose of, cf. §3. (δ) follows from (γ) by a known argument which will be given in §6. So the main difficulty lies in establishing the character of (γ), which will be done in §5. The discussion of (γ) and (δ)

would be much easier if the notion of being purely algebraic did not exclude the use of operators not in **M**, cf. §4.

§3. We begin by considering (α) and (β).

LEMMA 1. *$A \in \mathbf{M}$ is definite if and only if $A = B^*B$ for some $B \in \mathbf{M}$.*

PROOF. Sufficiency: Suppose $A = B^*B$ for $B \in \mathbf{M}$. Since $B^* \in \mathbf{M}$, $A \in \mathbf{M}$. Furthermore $(Af, f) = (B^*Bf, f) = (Bf, Bf) = \| Bf \|^2 \geqq 0$ and hence A is definite.

Necessity: If $A \in \mathbf{M}$ is definite, then there exists a definite $B \in \mathbf{M}$ with $B^2 = A$. (For the existence of B cf. [2], p. 307, Theorem 7, or [4], p. 142, §4.4. B is bounded along with A.[1] That $A \in \mathbf{M}$ implies $B \in \mathbf{M}$ is shown in [4], p. 143, Lemma 4.4.1. There exist also simple direct proofs, using functions of functional operators.) Now $B = B^*$ so $A = B^*B$.

LEMMA 2. *$|||A|||$ is the smallest (real) number $\alpha \geqq 0$ such that $\alpha^2 \cdot 1 - A^*A$ is definite.*

$|||A|||$ is the smallest $\alpha \geqq 0$ such that for every f, $\| Af \|^2 \leqq \alpha^2 \cdot \| f \|^2$. The inequality may be rewritten $0 \leqq \alpha^2 \cdot \| f \|^2 - \| Af \|^2 = \alpha^2(f, f) - (Af, Af) = \alpha^2(f, f) - (A^*Af, f) = (\{\alpha^2 \cdot 1 - A^*A\}f, f)$. Thus the statement, "for every f, $\| Af \|^2 \leqq \alpha^2 \cdot \| f \|^2$" is equivalent to "$\alpha^2 \cdot 1 - A^*A$ is definite." Substituting in the first sentence yields the lemma.

Thus we have shown

THEOREM I. *The notions (α) and (β) (definiteness and bound) are purely algebraical.*

§4. Let us now interrupt our discussion in order to analyze (γ) and (δ) without the necessity of avoiding the use of operators not in **M**. This is much easier than the discussion with the original observance, to be given in the two next sections.

$\lim_{n\to\infty}$ strong $A_n = 0$ means $\lim_{n\to\infty} \| A_n f \| = 0$ for all $f \in \mathfrak{H}$; $\lim_{n\to\infty}$ weak $A_n = 0$ means $\lim_{n\to\infty} | (A_n f, g) | = 0$ for every f and g in $\mathfrak{H}$. Clearly we may restrict ourselves in both cases to the f and g with $\| f \| = \| g \| = 1$.

Denote the closed linear set of all αf as usual by $[f]$, and its projection by $P_{[f]}$. Then one sees immediately that (for $\| f \| = \| g \| = 1$)

$$AP_{[f]}h = (f, h)Af, \qquad P_{[g]}AP_{[f]}h = (f, h) \cdot (Af, g)g.$$

Hence

$$|||AP_{[f]}||| = \| Af \|, \qquad |||P_{[g]}AP_{[f]}||| = | (Af, g) |.$$

Now the $P_{[f]}$ are obviously the minimal projections of the ring **B** of all bounded operators in $\mathfrak{H}$, i.e. those projections E for which the only projections $F \leqq E$

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[1] We have $\| Bf \|^2 = (B^*Bf, f) = (Af, f) \leqq \| Af \| \cdot \| f \| \leqq |||A||| \cdot \| f \|^2$. Thus $\| Bf \| \leqq \sqrt{|||A|||} \cdot \| f \|$. Hence B is bounded along with A.

are $F = 0, E$. (Çf. [4], pp. 143, 144, Definition 5.1.2. The assertion concerning minimal projections is obvious.) So we see

$$\lim_{n\to\infty} \text{strong } A_n = 0 \quad \text{means that always} \quad \lim_{n\to\infty} |||A_nE||| = 0$$

$$\lim_{n\to\infty} \text{weak } A_n = 0 \quad \text{means that always} \quad \lim_{n\to\infty} |||GA_nE||| = 0.$$

Here E, G run over all minimal projections of the ring $\mathbf{B}$ of all bounded operators in $\mathfrak{H}$.

The drawback in all this is that we had to refer to the ring $\mathbf{B}$ instead of $\mathbf{M}$.

§5. We now proceed to the more difficult analysis of (γ) in the original sense.

DEFINITION. *A sequence* $A_1, A_2, \cdots \in \mathbf{M}$ *is a* Σ *sequence if it possesses the properties*

(i) *The (numerical) sequence* $|||A_1|||, |||A_2|||, \cdots$ *is bounded.*

(ii) *There exists an operator* $X \in \mathbf{M}$ *such that* 1.) *for all* $C \in \mathbf{M}$ $CX = 0$ *implies* $C = 0$, *and* 2.) *all the operators,*

$$1 - \sum_{m=1}^{n} (A_mX)^*(A_mX), \qquad (n = 1, 2, \cdots) \quad \textit{are definite.}$$

LEMMA 3. *For every* Σ *sequence,* $A_1, A_2, \cdots \in \mathbf{M}$ *we have* $\lim_{n\to\infty}$ strong $A_n = 0$.

PROOF. For every $f \in \mathfrak{H}$ we have

$$(\{1 - \sum_{m=1}^{n} (A_mX)^*(A_mX)\}f, f) \geqq 0,$$

i.e.

$$\sum_{m=1}^{n} ((A_mX)^*(A_mX)f, f) \leqq (f, f),$$

i.e.

$$\sum_{m=1}^{n} ||A_mXf||^2 \leqq ||f||^2.$$

Consequently $\sum_{m=1}^{\infty} ||A_mXf||^2 \leqq ||f||^2$ and therefore $\lim_{m\to\infty} ||A_mXf|| = 0$.

Thus we have shown: The set $\mathfrak{S}$ of all $g \in \mathfrak{H}$ with $\lim_{m\to\infty} ||A_mg|| = 0$ contains the range of X.

$\mathfrak{S}$ is clearly a linear set, and since the $A_1, A_2, \cdots$ are uniformly bounded (by (i)), $\mathfrak{S}$ is closed. Let E be the projection on $\mathfrak{S}$. We next show $\mathfrak{S} \eta \mathbf{M}$. (For this notation, cf. [4], p. 141, Def. 4.2.1.) For suppose $U' \in \mathbf{M}'$ is unitary and $f \in \mathfrak{S}$. Then $\lim_{m\to\infty} ||A_mU'f|| = \lim_{m\to\infty} ||U'A_mf|| = \lim_{m\to\infty} ||A_mf|| = 0$. Thus $f \in \mathfrak{S}$ implies $U'f \in \mathfrak{S}$ and $\mathfrak{S}$ is invariant under every $U' \in \mathbf{M}'$ or $\mathfrak{S} \eta \mathbf{M}$. This implies $E \in \mathbf{M}$.

Since the range of X is contained in $\mathfrak{S}$, $EX = X$ or $(1 - E)X = 0$. Hence by (ii), $1 - E = 0$ and $1 = E$. Thus $\mathfrak{S} = \mathfrak{H}$ that is $\lim_{m\to\infty}$ strong $A_m = 0$.

LEMMA 4. *For every sequence,* $A_1, A_2, \cdots \in \mathbf{M}$ *with* $\lim_{m\to\infty}$ strong $A_n = 0$ *there exists a subsequence* $A_{1'}, A_{2'}, \cdots$ *which is a* Σ *sequence.*

PROOF. $|||A_1|||, |||A_2|||, \cdots$ is bounded by [1], p. 382, footnote 35), hence $|||A_{1'}|||, |||A_{2'}|||, \cdots$ is a *fortiori* bounded for every subsequence.

Consider now an everywhere dense sequence $f_1^0, f_2^0, \cdots$ in $\mathfrak{H}$.

For every $i = 1, 2, \cdots$, $\lim_{n\to\infty} ||A_nf_i^0|| = 0$. Choose accordingly $k(i)$ such

that for $n \geqq k(i)$, $\| A_n f_j^0 \| \leqq 1/2^i$ for all $j = 1, \cdots, i$. Choose a subsequence $1', 2', \cdots$ of $1, 2, \cdots$ with $1' < 2' < \cdots$ and $i' \geqq k(i)$. Then $\| A_{n'} f_j^0 \| \leqq 1/2^n$ if $n \geqq j$. Consequently $\sum_{n=1}^{\infty} \| A_{n'} f_j^0 \|^2$ is finite for $j = 1, 2, \cdots$.

Thus: The set $\mathfrak{F}$ of all $g \epsilon \mathfrak{H}$ for which $\sum_{n=1}^{\infty} \| A_{n'} g \|^2$ is finite, contains all $f_1^0, f_2^0, \cdots$ and therefore it is everywhere dense in $\mathfrak{H}$.

Form the space $\infty \otimes \mathfrak{H}$ of all sequences $\langle f_1, f_2, \cdots \rangle$ of elements of $\mathfrak{H}$ with $\sum_{i=1}^{\infty} \| f_i \|^2 < \infty$. $\infty \otimes \mathfrak{H}$ is a Hilbert space. (Cf. [4], pp. 136–137, §2.4.) Consider the following operator T from $\mathfrak{H}$ to $\infty \otimes \mathfrak{H}$.

$$Tf = \langle f, A_1 f, A_2 f, \cdots \rangle$$

(Cf. [3], Def. 1.2, p. 303.) The domain of this T is obviously the above defined set $\mathfrak{F}$. T is linear and it is also easy to verify that T is closed.[2] Therefore we may apply an argument for operators between Hilbert spaces similar to that of [2], p. 309, Satz 10, according to which there exists one and only one self-adjoint (i.e. hypermaximal) definite operator B in $\mathfrak{H}$ which is metrically equivalent to T, i.e. such that it has the same domain as T and always $\| Bf \| = \| Tf \|$. (Cf. [3], §4, pp. 311, 312.)

Thus the domain of B is $\mathfrak{F}$ and always

$$\| Bf \|^2 = \| f \|^2 + \sum_{i=1}^{\infty} \| A_{i'} f \|^2. \tag{1}$$

Consider a unitary $U' \epsilon \mathbf{M}'$. Then $U' A_{i'} = A_{i'} U'$ and hence (1) implies $\| BU'f \| = \| Bf \|$ and $\| U'^{-1} BU'f \| = \| Bf \|$. Thus $U'^{-1} BU'$ possesses the above properties which characterize B uniquely. So $U'^{-1} BU' = B$, i.e. $U'B = BU'$. Thus $B \eta \mathbf{M}$. (Cf. again [4], p. 141, Def. 4.2.1.)

By (1), $Bf = 0$ implies $f = 0$. This and the self-adjointness of B imply that B^{-1} is also self-adjoint. By (1), $\| Bf \| \geqq \| f \|$, hence $\| B^{-1} g \| \leqq \| g \|$ and so B^{-1} is bounded. Now $B \eta \mathbf{M}$ yields $B^{-1} \eta \mathbf{M}$ and since B^{-1} is bounded, $B^{-1} \epsilon \mathbf{M}$. (Cf. [4], p. 141, Lemma 4.2.1.) Put $X = B^{-1}$. Thus $X \epsilon \mathbf{M}$. Since $X = B^{-1}$, $CX = 0$ implies $Cf = CXBf = (CX) \cdot Bf = 0$ for f in the domain of B which is dense. For C bounded, this implies $C = 0$ and thus we have (ii) 1 of the definition of this section.

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[2] We must prove $\lim_{n \to \infty} \| f_n - f \| = 0$, $\lim_{n \to \infty} \| Tf_n - f^* \| = 0$ imply the existence of Tf and $Tf = f^*$.

Put $f^* = \langle f^{(0)}, f^{(1)}, f^{(2)}, \cdots \rangle$. Then $\lim_{n \to \infty} \| Tf_n - f^* \|^2 = 0$ means that

$$\lim_{n \to \infty} (\| f_n - f^{(0)} \|^2 + \sum_{i=1}^{\infty} \| A_{i'} f_n - f^{(i)} \|^2) = 0.$$

Hence, *a fortiori*

$$\lim_{n \to \infty} \| f_n - f^{(0)} \| = 0, \qquad \lim_{n \to \infty} \| A_{i'} f_n - f^{(i)} \| = 0.$$

Thus $f = f^{(0)}$ and $A_{i'} f = f^{(i)}$. Consequently

$$\| f \|^2 + \sum_{i=1}^{\infty} \| A_{i'} f \|^2 = \| f^0 \|^2 + \sum_{i=1}^{\infty} \| f^{(i)} \|^2 = \| f^* \|^2$$

i.e. $\sum_{i=1}^{\infty} \| A_{i'} f \|^2$ is finite.

So Tf is defined and $Tf = \langle f, A_{1'} f, A_{2'} f, \cdots \rangle = \langle f^{(0)}, f^{(1)}, f^{(2)}, \cdots \rangle = f^*$ as desired.

By equation (1), $\sum_{m=1}^{n} \| A_{m'} f \|^2 \leqq \| Bf \|^2$, hence $\sum_{m=1}^{n} \| A_{m'} Xg \|^2 \leqq \| g \|^2$. This may be written

$$\sum_{m=1}^{n} ((A_{m'} X)^*(A_{m'} X)g, g) \leqq (g, g)$$

or

$$(\{1 - \sum_{m=1}^{n} (A_{m'} X)^*(A_{m'} X)\}g, g) \geqq 0.$$

Thus the operator $1 - \sum_{m=1}^{n} (A_{m'} X)^*(A_{m'} X)$ is definite, i.e. we have (ii)2. This completes the proof.

LEMMA 5. *For a sequence* $A_1, A_2, \cdots, \epsilon \mathbf{M}$ *we have* $\lim_{n\to\infty}$ strong $A_n = 0$ *if and only if every subsequence* $A_{\bar{1}}, A_{\bar{2}}, \cdots$ *of* $A_1, A_2, \cdots$ *possesses a subsequence which is a* Σ*-sequence.*

PROOF: Necessity: $\lim_{n\to\infty}$ strong $A_n = 0$ implies for every subsequence $A_{\bar{1}}, A_{\bar{2}}, \cdots$ that $\lim_{n\to\infty}$ strong $A_{\bar{n}} = 0$. Hence Lemma 4 applies to $A_{\bar{1}}, A_{\bar{2}}, \cdots$ and so it has a subsequence $A_{1'}, A_{2'}, \cdots$ which is a Σ-sequence.

Sufficiency: Assume that we do not have $\lim_{n\to\infty}$ strong $A_n = 0$. Then there exists an $f \epsilon \mathfrak{H}$ for which we do not have $\lim_{n\to\infty} \| A_n f \| = 0$. Consequently there exists a subsequence $A_{\bar{1}}, A_{\bar{2}}, \cdots$ of $A_1, A_2, \cdots$ with $\lim_{n\to\infty} \| A_{\bar{n}} f \| = \alpha$ for an $\alpha \neq 0$ (but possibly $\alpha = \infty$). Thus for every subsequence, $A_{1'}, A_{2'}, \cdots$ of $A_{\bar{1}}, A_{\bar{2}}, \cdots$ equally $\lim_{n\to\infty} \| A_{n'} f \| = \alpha$ thus excluding $\lim_{n\to\infty}$ strong $A_{n'} = 0$. Hence Lemma 3 excludes that $A_{\bar{1}}, A_{\bar{2}}, \cdots$ be a Σ sequence.

Therefore, if the condition of our lemma is satisfied, we must have $\lim_{n\to\infty}$ strong $A_n = 0$.

Replacing $A_1, A_2, \cdots$ by $A_1 - A, A_2 - A, \cdots$ we can conclude from Lemma 5,

THEOREM II. *The notion* (γ) (*strong convergence*) *is purely algebraical.*

§6. (δ) can be deduced from (γ) by a known argument, which, nevertheless, we will give in full for the sake of completeness.

LEMMA 6. *If* $\lim_{n\to\infty}$ weak $A_n = A$ *there exists a subsequence* $A_{1'}, A_{2'}, \cdots$ *of* $A_1, A_2, \cdots$ *such that* $\lim_{n\to\infty}$ strong $\frac{1}{n} \sum_{m=1}^{n} A_{m'} = A$.

PROOF. Since we may replace $A, A_1, A_2, \cdots$ by $0, A_1 - A, A_2 - A, \cdots$ there is no loss of generality if we assume that $A = 0$, i.e. $\lim_{n\to\infty}$ weak $A_n = 0$.

By [1], p. 382, footnote 35, $|||A_1|||, |||A_2|||, \cdots$ is bounded. Let α be a bound, i.e. $|||A_n||| \leqq \alpha < \infty$.

Consider now an everywhere dense sequence $f_1^0, f_2^0, \cdots$ in $\mathfrak{H}$.

We shall define a subsequence $1', 2', \cdots$ of $1, 2, \cdots$ by induction. Assume therefore that $1', 2', \cdots, (m-1)'$ are already defined. We shall now define m'.

Since $\lim_{n\to\infty}$ weak $A_n = 0$, $\lim_{n\to\infty} (A_n f, g) = 0$ for any two f and $g \epsilon \mathfrak{H}$. Hence, in particular $\lim_{n\to\infty} (A_n f_i^0, A_{l'} f_i^0) = 0$ for all $l = 1', 2', \cdots, (m-1)'$ and all $i = 1, \cdots, m$. Consequently there exists a $k(m)$ such that $n \geqq k(m)$ implies $|(A_n f_i^0, A_{l'} f_i^0)| \leqq 1/2^m$ for all $l' = 1', \cdots, (m-1)'$ and all $i = 1, \cdots, m$. Now choose $m' \geqq k(m)$ and $> (m-1)'$.

Thus $1' < 2' < \cdots$ and $|(A_{m'}f_i^0, A_{l'}f_i^0)| \leqq 1/2^m$ for $m > l$ and $m \geqq i$. Interchanging m, l [3] gives $|(A_{m'}f_i^0, A_{l'}f_i^0)| \leqq 1/2^l$, for $m < l$ and $l \geqq i$. Summing up:

$$\{|(A_{m'}f_i^0, A_{l'}f_i^0)| \leqq 1/2^{\mathrm{Max}(m,l)} \quad \text{if } m \neq l \text{ and } \mathrm{Max}(m, l) \geqq i. \tag{2}$$

Now $n \geqq i$ yields

$$\begin{aligned}
\|(\textstyle\sum_{m=1}^n A_{m'})f_i^0\|^2 &= (\textstyle\sum_{m=1}^n A_{m'}f_i^0, \sum_{m=1}^n A_{m'}f_i^0) \\
&= \textstyle\sum_{m,l=1}^n (A_{m'}f_i^0, A_{l'}f_i^0) \\
&= \textstyle\sum_{m,l=1}^{i-1} (A_{m'}f_i^0, A_{l'}f_i^0) + \sum_{m=i}^n (A_{m'}f_i^0, A_{m'}f_i^0) \\
&\qquad + \textstyle\sum_{m,l=1(m\neq l,\mathrm{Max}(m,l)\geqq i)}^n (A_{m'}f_i^0, A_{l'}f_i^0) \\
&\leqq (i-1)^2\alpha^2\|f_i^0\|^2 + (n-i+1)\alpha^2\|f_i^0\|^2 \\
&\qquad + \textstyle\sum_{m,l=1(m\neq l,\mathrm{Max}(m,l)\geqq i)}^n 1/2^{\mathrm{Max}(m,l)} \\
&= \alpha^2\cdot\|f_i^0\|^2(n + (i-1)(i-2)) + \textstyle\sum_{k=i}^n \frac{2k-2}{2^k} \text{ [4]} \\
&\leqq \alpha^2\|f_i^0\|^2(n + (i-1)(i-2)) + 3 \text{ [5]}
\end{aligned}$$

and so

$$\left\|\left(\frac{1}{n}\sum_{m=1}^n A_{m'}\right)f_i^0\right\| \leqq \frac{1}{n}\sqrt{\|f_i^0\|^2\alpha^2[n + (i-1)(i-2)] + 3}$$

This implies $\lim_{n\to\infty}\left\|\left(\frac{1}{n}\sum_{m1}^n A_{m'}\right)f_i^0\right\| = 0.$

Thus we have shown: The set $\mathfrak{S}$ of all $g \in \mathfrak{H}$ with

$$\lim_{n\to\infty}\left\|\left(\frac{1}{n}\sum_{m=1}^n A_{m'}\right)g\right\| = 0 \quad \text{contains all } f_1^0, f_2^0, \cdots.$$

Hence $\mathfrak{S}$ is everywhere dense in $\mathfrak{H}$. Since all $|||A_{m'}||| \leqq \alpha\left|\left|\left|\frac{1}{n}\sum_{m=1}^n A_{m'}\right|\right|\right| \leqq \alpha$ for $n = 1, 2, \cdots$. Thus the $\frac{1}{n}\sum_{m=1}^n A_{m'}$ are uniformly bounded and consequently $\mathfrak{S}$ is a closed set. So $\mathfrak{S} = \mathfrak{H}$ and $\lim_{n\to\infty}$ strong $\frac{1}{n}\sum_{m=1}^n A_{m'} = 0$.

This completes the proof.

LEMMA 7. $\lim_{n\to\infty} A_n = A$ *if and only if every subsequence* $A_1, A_2, \cdots$ *of* $A_1, A_2, \cdots$ *possesses a subsequence* $A_{1'}, A_{2'}, \cdots$ *such that*

$$\lim_{n\to\infty}\text{ strong }\frac{1}{n}\sum_{m=1}^n A_{m'} = A.$$

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[3] Remember that $(A_{m'}f_i^{(0)}, A_{l'}f_i^{(0)}) = (A_{l'}f_i^{(0)}, A_{m'}f_i^{(0)})$.

[4] We introduce a new summation variable, $k = \mathrm{Max}(i, j)$. Given k the number of paris i, j which belong to it is clearly $2k - 2$.

[5] We have $\sum_{k=1}^\infty \frac{2k-2}{2^k} \leqq \sum_{k=1}^\infty \frac{2k-1}{2^k} = 3$.

PROOF. Necessity: Assume $\lim_{n\to\infty}$ weak $A_n = A$. For every subsequence $A_{\bar{1}}, A_{\bar{2}}, \cdots$ of $A_1, A_2, \cdots$, $\lim_{n\to\infty}$ weak $A_{\bar{n}} = A$. Hence Lemma 6 applies to $A_{\bar{1}}, A_{\bar{2}}, \cdots$ and so it has a subsequence $A_{1'}, A_{2'}, \cdots$ such that $\lim_{n\to\infty}$ strong $\frac{1}{n}\sum_{m=1}^{n} A_{m'} = A$. Thus the necessity of the hypothesis is established.

Sufficiency: Assume that the hypothesis is true but $\lim_{n\to\infty}$ weak $A_n = A$ is not true. Then there exists an f and $g \in \mathfrak{H}$ for which we do not have $\lim_{n\to\infty} (A_n f, g) = (Af, g)$. Consequently there exists a subsequence $A_{1^*}, A_{2^*}, \cdots$ of $A_1, A_2, \cdots$ with $\lim_{n\to\infty} (A_{n^*} f, g) = \alpha$ for an $\alpha \neq (Af, g)$ (but possibly $\alpha = \infty$). Hence if $A_{1'}, A_{2'}, \cdots$ is any subsequence of $A_{1^*}, A_{2^*}, \cdots$

$$\lim_{n\to\infty} (A_{n'} f, g) = \alpha \quad \text{and} \quad \alpha = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} (A_{m'} f, g)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{m=1}^{n} A_{m'} f, g\right).$$

On the other hand, by hypothesis there is a subsequence $A_{1'}, A_{2'}, \cdots$ of $A_{1^*}, A_{2^*}, \cdots$ such that $\lim_{n\to\infty}$ strong $\frac{1}{n}\sum_{m=1}^{n} A_{m'} = A$ and consequently $\lim_{n\to\infty} \left(\frac{1}{n}\sum_{m=1}^{n} A_{m'} f, g\right) = (Af, g) \neq \alpha$. This contradicts the last result of the preceding paragraph, which states that for every such sequence, the limit is α.

This contradiction proves the sufficiency of the hypothesis.

With Lemma 7, we have shown

THEOREM III. *The notion* (δ) (*weak convergence*) *is purely algebraical.*

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ON RINGS OF OPERATORS. IV

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INTRODUCTION

§1. The object of this paper is the investigation of the isomorphism properties of those operator rings which are *factors* and which are the substratum of several earlier publications of the authors. (Cf. [5], [6] and [8]. For the detailed definitions and also for the cases of factors—of which there will be more to say below—cf. [5], p. 138, Definition 3.1.2, p. 172, Theorem VIII.) The discrete cases—i.e. the cases (I)—have been exhaustively dealt with before. (Cf. [5], p. 173, Lemma 8.6.1, p. 139, Definitions 3.2.1 and 3.2.2 and Lemmas 3.2.1-3.2.3.) The purely infinite case—i.e. the case (III)—is the most refractory of all and we have, at least for the time being, scarcely any tools to investigate it. ([8] deals mainly with these factors.) Thus we are left with the continuous cases—i.e. the cases (II)—and they are our main objective in this paper.

An added justification of this program may be found in the fact that among all factors those of the finite continuous case—case (II_1)—have the strongest immediate interest. (Cf. [5], part V, [6] Chapter IV and Appendix.)

It will be seen however that the discrete cases can be included in our discussion with scarcely any extra effort. So we shall deal with them, i.e. with all cases but the purely infinite one.

For the discrete and continuous cases, the finite ones—i.e. the cases (I_n) ($n = 1, 2, \cdots$) and (II_1) respectively—are basic because the infinite ones—i.e. the cases (I_∞) and (II_∞) respectively—can be subsequently described with their help. (Cf. Theorem IX and Lemma 3.1.6) Therefore we shall direct our main effort on the finite cases. And since the discrete ones—(the cases (I_n), ($n = 1, 2, \cdots$))—are just the finite order matrix rings, this means essentially the above-mentioned continuous finite case (II_1).

§2. Let us now state the main problems of isomorphism more precisely.

Consider an operator ring $\mathbf{M}$ in a Hilbert space $\mathfrak{H}$ which contains 1. (We do not restrict $\mathbf{M}$ in any other way yet. For the significant definitions, cf. [2], pp. 388–389, Definitions 1–3). Then there exist two kinds of notions in $\mathbf{M}$: First, those which can be expressed in terms of the entity 1 (the unit operator) and the operations αA (α any complex number), A^*, $A + B$, $A \cdot B$ alone and referring only to the operators belonging to $\mathbf{M}$; Second those which need other things as well, e.g. operators outside of $\mathbf{M}$, elements of $\mathfrak{H}$, etc. The former notions are *purely algebraical*, the latter ones are not; we shall also call them spatial.

These notions were already investigated by the second author in [9].

It seems worth while to formulate this distinction in terms of isomorphisms.

Let, for each $i = 1, 2$ a Hilbert space $\mathfrak{H}_i$ and an operator ring $\mathbf{M}_i$ in $\mathfrak{H}_i$ be given.

We now define

A) Spatial isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$. This is a one-to-one isomorphic mapping of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ (i.e. one which is linear and isometric—unitary) which carries $\mathbf{M}_1$ into $\mathbf{M}_2$.

B) Algebraical ring isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$. This is a one-to-one mapping of $\mathbf{M}_1$ onto $\mathbf{M}_2$ which leaves the entity 1 and the operations αA (α any complex number) A^*, $A + B$, AB (when passing from $\mathbf{M}_1$ to $\mathbf{M}_2$) invariant.

(Cf. also [5], p. 145, §5.2, in particular Definition 5.2.1. We have added the requirement that 1 be invariant.)

Any spatial property is invariant under the *spatial isomorphisms* of A), while the purely algebraical properties are characterized by their invariance under the *algebraical isomorphisms* of B).

Clearly A) implies B) while the converse need not be true.

An important ring theoretical notion which is only spatial is $\mathbf{M}'$. (Cf. [2], p. 388, Def. 3. For the spatial character, cf. the detailed discussion of §3.3.) Nevertheless the factor property, i.e. $\mathbf{M} \cdot \mathbf{M}' = (\alpha \cdot 1)$ (cf. §3.3, loc. cit., and also the beginning of §1) is purely algebraical since it states that the operators $\alpha \cdot 1$ exhaust the *center* of $\mathbf{M}$ (Cf. [5], p. 138, Def. 3.1.2.)

Now our program is subdivided as follows.

Question I: When does B) hold?

Question II: Under what additional conditions does B) imply A)?

Question II was already investigated in a special case in [6]. It was shown there that under certain conditions A) and B) are equivalent. (Cf. [6], p. 244, Theorem XI.) We shall obtain a complete answer to Question II in Theorem X.

Question I is more difficult. It coincides with Problem 6 in [5], p. 172, and the present paper contains what progress the authors have been able to make in that direction. The main results are these: An extensive class of factors of case (II_1) which are all isomorphic to each other in the sense B) will be determined. These factors are called "approximately finite". (Cf. Theorems XII, XIV, which are based on Defs. 4.1.1, 4.3.1, 4.5.2 and 4.6.1 below.) The isomorphisms announced in [5], p. 229, (v) are contained in this class. On the other hand, certain factors of case (II_1) which are not isomorphic to the approximately finite ones will be constructed. (Cf. Theorems XVI, XVI′.)

§3. There are indications to the effect that the approximately finite factors are the simplest among those of case (II_1) but the evidence is not quite conclusive. It is true that every factor in case (II_1) has an approximately finite subring. (Cf. Theorem XIII). However, this "imbedding theorem" does not settle the matter, since the analogue of the Cantor-Bernstein "equivalence theorem" is not true: Two factors in the case (II_1) might be such that each is isomorphic to a sub-ring of the other, but the factors themselves may not be isomorphic. (An example of this is given in the appendix.) Hence the possibility exists that any factor in the case (II_1) is isomorphic to a sub-ring of any other such factor.

§4. The best we can do at present concerning the isomorphism problem in the case (II_1) is this: The limits of the approximate finite case are extended rather far in §5.2 and §5.6. The existence of non-approximately-finite factors in this case is established in Theorems XVI, XVI′. Certain algebraical invariants of factors in the case (II_1) are formed. ((1), (2), in §4.6, and the property Γ in Def. 6.1.1) of which the first two are probably of greater general significance, but the last one has so far been put to greater practical use. (Cf. the remark at the beginning of §6.1.)

The isomorphism questions of the case (II_∞) are reduced to those of the case (II_1). (This follows from Theorem IX.) Those of the discrete cases—the cases (I)—have been settled before (cf. above at the beginning of §1; also α) in Theorem IX).

This enumeration exhausts our present program. It answers Problems 6, 7 in [5], p. 172, and [v] eod., p. 229, as far as possible at this moment.

§5. We add that §5.3 contains a new technique of constructing factors in the case (II_1). This seems to be considerably simpler than our previous procedures, but it is closely related to them. (Cf. §5.4, §5.5.) It also throws some light on the meaning of this work from the point of view of the unitary representation theory of groups (cf. the remarks after Lemma 5.3.4). Further generalizations are probably possible and important. (Cf. the remark at the end of §5.6.)

The result that not all factors in the case (II_1) are isomorphic to each other, expressed in Theorem XVI′ deserves some further comment. From the point of view of the systematic build-up of the paper, it seemed best to put it at the end. It is, however, intelligible and of interest in itself, and it can be derived independently of most of the paper. The reader who is primarily interested in this particular result need only read §5.3, Def. 6.11, Lemma 6.1.1, Lemma 6.2.1, Lemma 6.2.2. The entire remainder of this paper is unnecessary for this purpose, in particular the extensive theories of algebraical and spatial types, of genera, and of approximate finiteness and its various equivalent forms. But we think, nevertheless, that knowledge of the whole paper will help to see this result too in the proper perspective.

§6. The notations are the same as in the papers referred to in the bibliography, particularly [5] and [6].

We use the Kronecker-Weierstrass symbol in the general sense:

$$\delta_{a,b} = \begin{cases} 1 \text{ for } a = b \\ 0 \text{ otherwise.} \end{cases}$$

for any objects a, b.

Bibliography

[1] John von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Annalen, vol. 102 (1929), pp. 49–131.

[2] John von Neumann, *Zur Algebra der Funktionaloperatoren*, Math. Annalen, vol. 102 (1929), pp. 370–427.

[3] JOHN VON NEUMANN, *Über Funktionen von Funktionaloperatoren*, Annals of Math., vol. 32 (1931), pp. 191–226.

[4] JOHN VON NEUMANN, *On a certain topology for rings of operators*, Annals of Math., vol. 37 (1936), pp. 111–115.

[5] F. J. MURRAY AND JOHN VON NEUMANN, *On rings of operators*, Annals of Math., vol. 37 (1936), pp. 116–229.

[6] F. J. MURRAY AND JOHN VON NEUMANN, *On rings of operators II*, Trans. Amer. Math. Socl, vol. 41 (1937), pp. 208–248.

[7] JOHN VON NEUMANN, *On infinite direct products*, Compositio Mathematica, vol. 6 (1938), pp. 1–77.

[8] JOHN VON NEUMANN, *On rings of operators III*, Annals of Math, vol. 41 (1940), pp. 94–161.

[9] JOHN VON NEUMANN, *On some algebraical properties of operator rings*, Annals of Math., vol. 44, preceding immediately.

CONTENTS

INTRODUCTION

1. Purpose of the paper.
2. Algebraical and spatial isomorphisms. Questions I, II.
3. Approximate finiteness. Imbedding.
4. Outline of results.
5. The role of new examples.
6. Notations.

CHAPTER I. THE METRIC OF **M**. PURELY ALGEBRAICAL CHARACTER OF CERTAIN NOTIONS

§1.1 Purely algebraical concepts; Difficulties; The operator topologies; Subrings.
§1.2 Character of $Tr_{\mathbf{M}}(A)$, $[[A]]$. Metric and restricted metric closure. Linear sets. Algebras.
§1.3 Restricted metric convergence and strong convergence.
§1.4 Metric closures and strong closure.
§1.5 Continuation of the discussion of closure.
§1.6 Conclusion of the discussion of closure.
Theorem I. Equivalence of various closures.
Theorem II. Subrings are purely algebraical concepts.

CHAPTER II. THE ALGEBRAICAL TYPE AND ITS OPERATIONS. THE FUNDAMENTAL GROUP

§2.1 Algebraical type $\bar{\mathbf{M}}$. Discussion.
§2.2 Conjugate dual isomorphisms. $\bar{\mathbf{M}}`$ and $\bar{\mathbf{M}}'$.
§2.3 Properties of the above. $\bar{\mathbf{M}}^c = \bar{\mathbf{M}}` = \bar{\mathbf{M}}'$.
Theorem III. Properties of $\bar{\mathbf{M}}^c$.
§2.4 p-matrix isomorphisms. Construction of $\bar{\mathbf{M}}^p$.
Theorem IV. Properties of $\bar{\mathbf{M}}^p$.
§2.5 One-valuedness of $\bar{\mathbf{M}}^\infty$.
Theorem IV′. One-valuedness of $\bar{\mathbf{M}}^\infty$.
§2.6 Inversion of $\bar{\mathbf{M}} = \bar{\mathbf{N}}^p$. Existence of $\bar{\mathbf{M}}^{1/p}$.
Theorem V. Inversion of $\bar{\mathbf{M}} = \bar{\mathbf{N}}^p$.
§2.7 The operation $\bar{\mathbf{N}}^p$ and the factor character, case, finiteness.
§2.8 The operation $\bar{\mathbf{M}}^\alpha$, $(0 < \alpha \leqq 1)$ and its properties
Theorem VI. Properties of $\bar{\mathbf{M}}^\alpha$, $(0 < \alpha \leqq 1)$
§2.9 The operation $\bar{\mathbf{M}}^\theta$, (θ general) and its properties
Theorem VII. Properties of $\bar{\mathbf{M}}^\theta$ (θ general)
§2.10 The fundamental group $\mathfrak{G}(\bar{\mathbf{M}})$. Special cases.
Theorem VIII. Properties of $\mathfrak{G}(\bar{\mathbf{M}})$.

CHAPTER III. CHARACTERIZATION OF ALL SPATIAL TYPES IN TERMS OF ALGEBRAICAL TYPES
§3.1 Commensurability. The genus $\overline{\overline{\mathbf{M}}}$.
§3.2 Elementary properties of the genus. Genus and fundamental group.
§3.3 The spatial type $\ddot{\mathbf{M}}$. The number $\theta = (1/c)(D_{\mathbf{M}'}(\mathfrak{H})/D_{\mathbf{M}}(\mathfrak{H}))$.
Characterization of $\ddot{\mathbf{M}}$ in terms of $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ and θ. Precise relationship between $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ and θ.
Theorem X, $\ddot{\mathbf{M}}$ and $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ and θ.
CHAPTER IV. APPROXIMATE FINITENESS.
§4.1 Approximate finiteness of type $[p_1, p_2, \cdots]$.
Theorem XI. First isomorphism theorem.
§4.2 Further approximation results.
§4.3 Approximate finiteness (A).
§4.4 Comparison of (A) and type $[p_1, p_2, \cdots]$.
§4.5 Approximate finiteness (B). Structure of finite rings.
§4.6 Approximate finiteness (C). Equivalence results.
Theorem XII. Equivalence theorem.
§4.7 Existence results.
Theorem XIII. Imbedding theorem.
Theorem XIV. Final isomorphism theorem.
§4.8 The operations $\overline{\mathbf{M}}^c$ and $\overline{\mathbf{M}}^\theta$ in conjunction with approximate finiteness. The corresponding invariants
Theorem XV. $\overline{\mathbf{A}}^c = \overline{\mathbf{A}}$, $\mathfrak{G}(\overline{\mathbf{A}}) = P$.
CHAPTER V. DISCUSSION OF VARIOUS EXAMPLES.
§5.1 General remarks on the examples.
§5.2 Approximate finiteness of certain existing examples.
§5.3 Construction of the new examples.
§5.4 First connection between the two kinds of examples.
§5.5 Second connection between the two kinds of examples.
§5.6 Approximate finiteness among the new examples.
CHAPTER VI. NOT APPROXIMATELY FINITE FACTORS.
§6.1 The property Γ and approximate finiteness.
§6.2 Existence of not approximately finite factors.
Theorem XVI. Existence theorem.
Theorem XVI'. Existence of several algebraical types of case (II_1).
§6.3 Discussion of further examples.
APPENDIX: Example of two non-isomorphic factors in Case (II_1), each of which is isomorphic to a subring of the other.
Theorem A. Incomparability result.

CHAPTER I. THE METRIC OF M

PURELY ALGEBRAICAL CHARACTER OF VARIOUS NOTIONS

§1.1 Consider a Hilbert space $\mathfrak{H}$ and an operator ring $\mathbf{M}$ in $\mathfrak{H}$. We know from §2 in the introduction as well as from [9] that the purely algebraical character of certain notions in $\mathbf{M}$ is neither obvious nor certain.

It was demonstrated in [9], however, that the following notions in $\mathbf{M}$ are purely algebraical.

Definiteness of an $A \in \mathbf{M}$; the numerical value $|||A|||$ of the bound of an $A \in \mathbf{M}$; convergence in the sense of the strong and of the weak operator topologies. (Convergence in the sense of the strongest topology is the same as in the strong one. Cf. [4], p. 112, end of §2.)

We did not establish the same thing for the operator topologies themselves,

i.e. for the strongest, the strong and the weak. (For the definitions, cf. [4], p. 111, §2, and [2], pp. 381–382.) The uniform topology, (cf. [2], pp. 384–386) causes no difficulty, since $|||\ A\ |||$ is purely algebraical, but it will play no role in our discussions. This is very unsatisfactory since the notion of a subring of **M** must be based on one of these topologies. (Cf. [4], p. 112, beginning of §3; [2], p. 396, end of §3; p. 388, Definition 1. Best consulted in the reverse order.) They cannot be replaced by convergence notions, nor by the uniform topology. (Cf. in particular [2], p. 382–3 and p. 384.) And yet the notion of a subring ought to be purely algebraical.

For our present purposes it will be adequate to settle these questions for a restricted class of rings **M** and this we now do.

§1.2 Observe first that the following notions are purely algebraical.

The factor character of **M** and the case to which **M** belongs, the numerical value of the relative dimension function $D_{\mathbf{M}}(E)$ (E any projection $\epsilon\,\mathbf{M}$) apart from its normalization and even the standard normalization in the cases where one is defined; the numerical value of the relative trace $Tr_{\mathbf{M}}(A)$, ($A\,\epsilon\,\mathbf{M}$) in the finite cases.

These assertions were established in [5], p. 145, Lemma 5.2.1; p. 173, Theorem IX; [6], p. 219, Property IV (since that characterization is purely algebraical).

We assume now that **M** is a factor in a finite case.[1] We shall use the relative dimension function $D_{\mathbf{M}}(E)$ and it will be advantageous not to normalize it. We shall also use the relative trace $Tr_{\mathbf{M}}(A)$ with its usual properties. We define for every $A\,\epsilon\,\mathbf{M}$

$$[[A]] = \sqrt{Tr_{\mathbf{M}}(A^*A)} = \sqrt{Tr_{\mathbf{M}}(AA^*)}. \tag{1.2.α}$$

(Cf. [6], p. 241, Lemma 4.3.2. The considerations of [8], p. 102, Def. 1.3.1 and Theorem II, are somewhat more general but of the same type.) Owing to our above remarks, the numerical value of $[[A]]$ is purely algebraical too.

By [6], p. 235, Theorem III, there exists a fixed finite system $g_1, \cdots, g_m\,\epsilon\,\mathfrak{H}$ (depending on **M** only), such that for all $A\,\epsilon\,\mathbf{M}$

$$Tr_{\mathbf{M}}(A) = \sum_{i=1}^{m}(Ag_i, g_i). \tag{1.2.β}$$

Combining (1.2.α) with (1.2.β), and remembering that $(A^*Ag, g) = ||\,Ag\,||^2$, $(AA^*g, g) = ||\,A^*g\,||^2$ we obtain

$$[[A]] = \sqrt{\sum_{i=1}^{m} ||\,Ag_i\,||^2} = \sqrt{\sum_{i=1}^{m} ||\,A^*g_i\,||^2}. \tag{1.2.γ}$$

This formula is useful for many purposes, although it obscures the purely algebraical character of $[[A]]$.

$[[A]]$ has all the properties of a *norm* in a linear space, and accordingly $[[A - B]]$ has all those of a *distance*. (Cf. for these and for some further properties [[6]], p. 242, Property II°. We need these for **M** only, and not, as loc. cit., for

[1] A finite case is either a (I_n), $n = 1, 2, \cdots$ or a (II_1). The inclusion of the former is unnecessary, since our real interest is in case (II_1). But these considerations have the totality of all finite cases as their natural scope.

its extension $Q(\mathbf{M})$. All these properties can be read off the representation (1.2.γ)—the decisive fact being, of course, that we have them both.)

Thus $\mathbf{M}$ has become a topological space in a new way, with a purely algebraical metric.[2]

We now define

DEFINITION 1.2.1. *A sequence* $A_1, A_2, \cdots \in \mathbf{M}$ *is metrically convergent to* $A \in \mathbf{M}$ *if* $\lim_{\nu\to\infty} [[A_\nu - A]] = 0$. *It is restrictedly metrically convergent to* $A \in \mathbf{M}$ *if* $\lim_{\nu\to\infty} [[A_\nu - A]] = 0$ *and the (numerical) sequence* $|||A_1|||, |||A_2|||, \cdots$ *is bounded.*

The notions of closure for subsets $\mathbf{S}$ of $\mathbf{M}$ which are induced by these two notions of convergence, are the *"metrical closure"* and the *"restricted metrical closure"* of $\mathbf{S}$.[3]

We shall also need:

DEFINITION 1.2.2. *A subset* $\mathbf{S}$ *of* $\mathbf{M}$ *is linear if* $A, B \in \mathbf{S}$ *imply* αA, $A + B \in \mathbf{S}$. *It is an algebra if* $A, B \in \mathbf{S}$ *imply* αA, A^*, $A + B$, $AB \in \mathbf{S}$.

Clearly both these definitions define purely algebraical notions.

Now a subring $\mathbf{S}$ of $\mathbf{M}$ is a closed algebra, where closure may be understood in any one of the three operator topologies: the strongest, the strong, or the weak (cf. the end of §1.1). We shall establish the purely algebraical character of the notion of a subring by proving the equivalence of the notion of closure in all these topologies to the purely algebraical closures of Def. 1.2.1 for a set $\mathbf{S}$ which is an algebra.[4]

§1.3 Observe first:

LEMMA 1.3.1. *For the* $g_1, \cdots, g_m$ *of* (1.2.β) *and* (1.2.γ) *the* $A'g_i$, ($A' \in \mathbf{M}'$, $i = 1, \cdots, m$) *span the closed linear set* $\mathfrak{H}$.

PROOF. For a fixed i ($= 1, \cdots, m$) the $A'g_i$, $A' \in \mathbf{M}'$ span the closed linear set $\mathfrak{M}_{g_i}^{\mathbf{M}'}$ (Cf. [5], p. 143, Def. 5.1.1.) Put $\mathfrak{N} = [\mathfrak{M}_{g_i}^{\mathbf{M}'}, i = 1, \cdots, m]$. We must prove $\mathfrak{N} = \mathfrak{H}$. Now $\mathfrak{M}_{g_i}^{\mathbf{M}'} \eta \mathbf{M}$ (cf. loc. cit., Lemma 5.1.1). Hence $\mathfrak{N} \eta \mathbf{M}$ and $E = P_{\mathfrak{N}} \in \mathbf{M}$. Also $g_i \in \mathfrak{M}_{g_i}^{\mathbf{M}'} \subset \mathfrak{N}$ so

$$(1.3.\alpha) \qquad Eg_i = g_i .$$

Now (1.2.γ) shows that if $A \in \mathbf{M}$ and $Ag_i = 0$ for $i = 1, \cdots, m$, then $A = 0$. Apply this to $A = 1 - E$. Then (1.3.α) yields $1 - E = 0$, $P_{\mathfrak{N}} = E = 1$ as desired.

---

[2] The uniform topology, already mentioned in §1.1, is actually of the same character: $|||A|||$, $|||A - B|||$ are a norm and a metric, just as $[[A]]$, $[[A - B]]$. But only the latter is useful in the theory of operator rings. This is due to its connection with the strong topology, which will appear in §1.4 and §1.5.

[3] The metrical closure is the conventional topological notion, based on the specified metric. The restricted metrical closure is not of that nature. Thus it is not evident that the set of all limits of all restrictedly metrically convergent sequences from a given set $\mathbf{S}$ is necessarily closed in that sense. Actually this is not true for all sets $\mathbf{S}$ although when $\mathbf{S}$ is an algebra it does follow from the results of §1.5 (cf. there).

[4] Observe that restrictions on both the ring $\mathbf{M}$ and its subset $\mathbf{S}$ are necessary in order to secure this result. $\mathbf{M}$ must be a factor in a finite case, $\mathbf{S}$, an algebra.

We now are able to prove:

LEMMA 1.3.2. *The sequence* $A_1, A_2, \cdots \in \mathbf{M}$ *is strongly convergent to* $A \in \mathbf{M}$ *if and only if it is restrictedly metrically convergent to this* A.

PROOF. Necessity: Assume that $A_1, A_2, \cdots$ is strongly convergent to A. Then the (numerical) sequence $|||A_1|||, |||A_2|||, \cdots$ is bounded by [2], p. 382, footnote 35, and $\lim_{\nu\to\infty} [[A_\nu - A]] = 0$ by (1.2.γ). Hence we have restricted metric convergence.

Sufficiency: Assume that $A_1, A_2, \cdots$ are restrictedly metrically convergent to A. We must show that they converge strongly too, i.e. that the set $\mathfrak{M}$ of all f with $\lim_{\nu\to\infty} ||(A_\nu - A)f|| = 0$ is $\mathfrak{H}$. $\mathfrak{M}$ is obviously a linear set. Since the (numerical) sequence $|||A_1|||, |||A_2|||, \cdots$ is bounded, $\mathfrak{M}$ is also closed. Hence it suffices to show that $\mathfrak{M}$ spans the closed linear set $\mathfrak{H}$.

Now by (1.2.γ), $||(A_\nu - A)g_i|| \leqq [[A_\nu - A]]$. Hence $\lim_{\nu\to\infty} ||(A_\nu - A)g_i|| = 0$. Therefore for every $A' \in \mathbf{M}'$, $\lim_{\nu\to\infty} ||A'(A_\nu - A)g_i|| = 0$. But $A'(A_\nu - A) = (A_\nu - A)A'$ since $A' \in \mathbf{M}'$ and $A_\nu, A \in \mathbf{M}$. Consequently $\lim_{\nu\to\infty} ||(A_\nu - A)A'g_i|| = 0$ and all $A'g_i \in \mathfrak{M}$. Hence $\mathfrak{M}$ spans the closed linear set $\mathfrak{H}$ by Lemma 1.3.1, and the proof is completed.

Thus closure with respect to strong convergence is equivalent to restricted metric closure for every subset $\mathbf{S}$ of $\mathbf{M}$. This however is not enough to establish the purely algebraical character of the notion of a subring, since that involves topological closure which cannot be replaced by convergence closure (cf. the end of §1.1). In fact, from that point of view we have made no progress at all, since we knew all along from [9] that strong convergence is a purely algebraical notion for all rings $\mathbf{M}$. Besides our present restrictions on $\mathbf{M}$, we shall also have to restrict $\mathbf{S}$ to algebras before we can get any further (cf. the end of §1.2).

§1.4 Consider first the following two simple results, valid for all subsets $\mathbf{S}$ of $\mathbf{M}$.

LEMMA 1.4.1. *Strong closure of* $\mathbf{S}$ *implies the restricted metric closure.*

PROOF. Strong closure of $\mathbf{S}$ implies closure with respect to strong convergence, and by Lemma 1.3.2 the latter is equivalent to restricted metric closure.

LEMMA 1.4.2. *Metric closure of* $\mathbf{S}$ *implies its strong closure.*

PROOF. Let $\mathbf{S}$ be metrically closed and consider a strong condensation point A of $\mathbf{S}$. Since $\mathbf{S} \subset \mathbf{M}$ and $\mathbf{M}$ is strongly closed, this implies $A \in \mathbf{M}$.

For every $\nu = 1, 2, \cdots$ choose an $A_\nu \in \mathbf{S}$ with $||(A_\nu - A)g_i|| \leqq 1/\nu$ for $i = 1, \cdots, m$. This can be done since, by the definition of a limit point, we must have at least one $A_\nu \in \mathbf{M}$ in the strong neighborhood $U_3(A, g_1, \cdots g_m; 1/\nu)$ of A (cf. [2], §2, p. 381). Then (1.2.γ) gives $[[A_\nu - A]] \leqq \sqrt{m}/\nu$ and hence $\lim_{\nu\to\infty} [[A_\nu - A]] = 0$. Thus $A_1, A_2, \cdots$ converves metrically to A and since $\mathbf{S}$ is metrically closed, $A \in \mathbf{S}$. This completes the proof.

Now if we can show that the restricted metric closure implies metric closure, the above lemmas will yield the equivalence of the strong, the metric and the restricted metric notions of closure. As pointed out at the end of §1.3, this is what we need. The assumption that $\mathbf{S}$ is an algebra will be needed to secure the above implication.[4]

§1.5 In what follows we shall make repeated use of the notion and properties of the functions of bounded Hermitian and of unitary operators,[5] and it will be convenient to omit specific references to that theory.

LEMMA 1.5.1. (i) *If A is bounded Hermitian, then $(A - i1)^{-1}$ can be formed and*

$$\frac{A + i1}{A - i1} = (A - i\cdot 1)^{-1}(A + i1) = (A + i1)(A - i1)^{-1}$$

is unitary.

(ii) If $A, B \in \mathbf{M}$ and Hermitian, then $\frac{A + i1}{A - i1}, \frac{B + i1}{B - i1} \in \mathbf{M}$ too, and

$$\left[\left[\frac{A + i1}{A - i1} - \frac{B + i1}{B - i1}\right]\right] \leqq 2[[A - B]].$$

PROOF. Ad. (i). Consider the two functions

$$f(\lambda) = \frac{1}{\lambda - i}, \qquad g(\lambda) = \frac{\lambda + i}{\lambda - i}.$$

Since they are continuous for all real λ we can form $f(A)$ and $g(A)$. We also have

$$(\lambda - i)f(\lambda) = f(\lambda)(\lambda - i) = 1$$
$$(\lambda + i)f(\lambda) = f(\lambda)(\lambda + i) = g(\lambda).$$

Therefore $f(A) = (A - i1)^{-1}$ and $g(A) = (A + i1)f(A) = f(A)(A + i1)$. Thus $g(A) = (A + i1)(A - i1)^{-1} = (A - i1)^{-1}(A + i1)$ or $g(A) = (A + i1)/(A - i1)$ in the sense indicated above.

Since $\overline{g(\lambda)}g(\lambda) = 1 = g(\lambda)\overline{g(\lambda)}$ for all real λ, $g(A)^*g(A) = 1 = g(A)g(A)^*$ and hence $g(A)$ is unitary. Thus the proof of (i) is completed. We note also that inasmuch as $|f(\lambda)| \leqq 1$ for all real λ, $|||f(A)||| \leqq 1$ or

$$|||(A - i1)^{-1}||| \leqq 1. \tag{1.5.α}$$

Ad (ii), $A, B \in \mathbf{M}$ imply $g(A), g(B) \in \mathbf{M}$ or $(A + i1)/(A - i1), (B + i1)/(B - i1) \in \mathbf{M}$.

We have

$$\begin{aligned}\frac{A + i1}{A - i1} - \frac{B + i1}{B - i1} &= (A - i1)^{-1}(A + i1) - (B + i1)(B - i1)^{-1} \\ &= (A - i\cdot 1)^{-1}\{(A + i1)(B - i1) \\ &\qquad - (A - i\cdot 1)(B + i\cdot 1)\}(B - i1)^{-1} \\ &= -2i(A - i1)^{-1}(A - B)(B - i\cdot 1)^{-1}.\end{aligned}$$

[5] Cf., e.g., [3], pp. 205 and 220. The real function-theory methods of [3] are actually much more than what is needed for the limited objectives of §1.5, as only continuous functions (sometimes even polynomials) of operators occur. ((i) in Lemma 1.5.1 can even be proven by elementary evaluations.) But it would take up too much space to go into such methodological questions systematically.

Consequently

$$\left[\left[\frac{A+i1}{A-i1}-\frac{B+i1}{B-i1}\right]\right] \leqq 2\,|||\,(A-i\cdot 1)^{-1}\,|||\,[[A-B]]\,|||\,(B-i\cdot 1)^{-1}\,|||$$

and by (1.5.α)

$$\left[\left[\frac{A+i1}{A-i1}-\frac{B+i1}{B-i1}\right]\right] \leqq 2[[A-B]].$$

LEMMA 1.5.2. *Let a function $\varphi(\xi)$ be given, defined and continuous for all ξ with $|\xi| = 1$. Then there exists a function $\omega_1(\epsilon)$ (given φ, ω_1 is determined) defined and >0 for all $\epsilon > 0$, with the following property: If U, V are unitary and $\epsilon \mathbf{M}$ then $\varphi(U), \varphi(V) \epsilon \mathbf{M}$ and $[[U-V]] < \omega_1(\epsilon)$ implies $[[\varphi(U)-\varphi(V)]] < \epsilon$.*

PROOF. That U, V unitary and $\epsilon \mathbf{M}$ imply $\varphi(U) \epsilon \mathbf{M}$, $\varphi(V) \epsilon \mathbf{M}$ is obvious.

We shall now show that it suffices to prove our assertion for all trigonometrical polynomials

$$\varphi(\xi) = \textstyle\sum_{\nu=-n}^{n} a_\nu \xi^\nu.$$ [6]

Assume accordingly that it is true for these and consider a general $\varphi(\xi)$.

Consider an $\epsilon' > 0$. Choose a trigonometrical polynomial

$$\varphi^{\epsilon'}(\xi) = \textstyle\sum_{\nu=-n^{\epsilon'}}^{n^{\epsilon'}} a_\nu^{\epsilon'} \xi^\nu$$

such that $|\varphi(\xi) - \varphi^{\epsilon'}(\xi)| \leqq \frac{1}{3}\epsilon'$ for all ξ with $|\xi| = 1$[6]. Then the unitary of U, V implies $|||\varphi(U) - \varphi^{\epsilon'}(U)||| \leqq \frac{1}{3}\epsilon'$ and $|||\varphi(V) - \varphi^{\epsilon'}(V)||| \leqq \frac{1}{3}\epsilon'$. Hence, *a fortiori*,

$$[[\varphi(U) - \varphi^{\epsilon'}(U)]] \leqq \tfrac{1}{3}\epsilon', \qquad [[\varphi(V) - \varphi^{\epsilon'}(V)]] \leqq \tfrac{1}{3}\epsilon'.$$

Denote the $\omega_1(\epsilon)$ of $\varphi^{\epsilon'}(\xi)$ by $\omega_1^{\epsilon'}(\epsilon)$. Then $[[U-V]] < \omega_1^{\epsilon'}(\frac{1}{3}\epsilon')$ implies

$$[[\varphi^{\epsilon'}(U) - \varphi^{\epsilon'}(V)]] < \tfrac{1}{3}\epsilon'$$

and so, owing to the above,

$$[[\varphi(U) - \varphi(V)]] < \epsilon'.$$

Thus $\omega_1(\epsilon') = \omega_1^{\epsilon'}(\frac{1}{3}\epsilon')$ meets our requirements for the original $\varphi(\xi)$.

It only remains therefore to prove our assertion for the trigonometrical polynomials

$$\varphi(\xi) = \textstyle\sum_{\nu=-n}^{n} a_\nu \xi^\nu.$$

Obviously we may even restrict ourselves to the monomials

$$\varphi(\xi) = \xi^\nu, \qquad (\nu = 0 \pm 1, \pm 2, \cdots).$$

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[6] Since $|\xi| = 1$ we may put $\xi = e^{i\alpha}$, α real, mod 2π. Now

$$\varphi(\xi) = \textstyle\sum_{\nu=-n}^{n} a_\nu \xi^\nu = \sum_{\nu=-n}^{n} a_\nu e^{i\nu\alpha}.$$

Thus these $\varphi(\xi)$ are the trigonometrical polynomials with their familiar approximation properties. Cf. Zygmund, Antoni, "Trigonometrical Series", Warsaw (1935), §3.23 (iii), p. 47.

For $\nu = 0$ this is trivial, and for $\nu < 0$ the unitarity of U, V implies $[[U^\nu - V^\nu]] = [[(U^\nu - V^\nu)^*]] = [[U^{*\nu} - V^{*\nu}]] = [[U^{-\nu} - V^{-\nu}]]$.

Therefore we may even assume $\nu > 0$. In this case, however,

$$\begin{aligned} U^\nu - V^\nu &= \textstyle\sum_{\mu=1}^{\nu}(U^\mu V^{\nu-\mu} - U^{\mu-1}V^{\nu-\mu+1}) \\ &= \textstyle\sum_{\mu=1}^{\nu} U^{\mu-1}(U - V)V^{\nu-\mu}. \\ [[U^\nu - V^\nu]] &= [[\textstyle\sum_{\mu=1}^{\nu} U^{\mu-1}(U - V)V^{\nu-\mu}]] \\ &= \textstyle\sum_{\mu=1}^{\nu} |||\, U^{\mu-1} \,|||\, [[U - V]] \,|||\, V^{\nu-\mu} \,||| \\ &= \textstyle\sum_{\mu=1}^{\nu} [[U - V]] = \nu[[U - V]]. \end{aligned}$$

Thus $\omega_1(\epsilon) = (1/\nu)\epsilon$ meets our requirements for these $\varphi(\xi)$.

LEMMA 1.5.3. *Let a function $\psi(\lambda)$ be given, defined and continuous for all real λ and for which $\lim_{\lambda\to\pm\infty}\psi(\lambda)$ exists.*[7] *Then there exists a function $\omega_2(\epsilon)$ (given ψ, ω is determined) defined and > 0 for all $\epsilon > 0$ with the following property: If A, B are Hermitian and $\epsilon \mathbf{M}$ then $\psi(A)$, $\psi(B) \epsilon \mathbf{M}$ and $[[A - B]] < \omega_2(\epsilon)$ implies $[[\psi(A) - \psi(B)]] < \epsilon$.*

PROOF. That A, B Hermitian and $\epsilon \mathbf{M}$ imply $\psi(A)$, $\psi(B) \epsilon \mathbf{M}$ is obvious.

The correspondence

$$\xi = \frac{\lambda + i}{\lambda - i} \quad \text{or equivalently} \quad \lambda = i\frac{\xi + 1}{\xi - 1}$$

is a one-to-one and bicontinuous mapping of ξ with $|\xi| = 1$ on all real λ and $\lambda = \pm\infty$ where $\xi = 1$ corresponds to $\lambda = \pm\infty$. Hence

$$\varphi(\xi) = \begin{cases} \psi\left(i\dfrac{\xi + 1}{\xi - 1}\right) & \text{for } |\xi| = 1,\ \xi \neq 1 \\ \lim_{\lambda\to\pm\infty}\psi(\lambda) & \text{for } \xi = 1 \end{cases}$$

is defined and continuous for all ξ with $|\xi| = 1$. Clearly

$$\psi(\lambda) = \varphi\left(\frac{\lambda + i}{\lambda - i}\right).$$

We recall the $g(\lambda) = (\lambda + i)/(\lambda - i)$ used in the proof of Lemma 1.5.1 and let $U = g(A) = (A + i\cdot 1)/(A - i\cdot 1)$, $V = g(B) = (B + i1)/(B - i1)$. Then the above relations imply

$$\psi(A) = \varphi(U), \qquad \psi(B) = \varphi(V).$$

By Lemma 1.5.1, U and V are unitary and $\epsilon \mathbf{M}$ and

$$[[U - V]] \leqq 2[[A - B]].$$

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[7] We require $\lim_{\lambda\to\infty}\psi(\lambda) = \lim_{\lambda\to-\infty}\psi(\lambda)$. This condition could be removed, but it simplifies the proof and suffices for our present purposes.

Now Lemma 1.5.2 applies to $\varphi(\xi)$. Consider the resulting $\omega_1(\epsilon)$. Then $[[A - B]] < \frac{1}{2}\omega_1(\epsilon)$ implies $[[U - V]] < \omega_1(\epsilon)$ and hence $[[\varphi(U) - \varphi(V)]] < \epsilon$ or $[[\psi(A) - \psi(B)]] < \epsilon$. Therefore $\omega_2(\epsilon) = \frac{1}{2}\omega_1(\epsilon)$ meets our requirements.

LEMMA 1.5.4. *Let an algebra* $\mathbf{S}$ *in* $\mathbf{M}$ *be given and a sequence* $A_1, A_2, \cdots \epsilon \mathbf{S}$ *which converges metrically to an* $A \epsilon \mathbf{M}$. *Then there exists another sequence,* $A_1', A_2', \cdots \epsilon \mathbf{S}$ *which converges restrictedly metrically to* A.

PROOF. $A_1, A_2, \cdots$ converges metrically to A; hence $A_1^*, A_2^*, \cdots$ converges metrically to A^*.[8] Consequently $(1/2)(A_1 + A_1^*), (1/2)(A_2 + A_2^*), \cdots$ converges metrically to $(1/2)(A + A^*)$, and $(1/2i)(A_1 - A_1^*), (1/2i)(A_2 - A_2^*), \cdots$ converges metrically to $(1/2i)(A - A^*)$. The operators occurring in the two last assertions are all Hermitian and $\epsilon \mathbf{S}$. Also $A = (1/2)(A + A^*) + i(1/2i)(A - A^*)$. Now suppose our result has been established in the special case in which all the operators $A_1, A_2, \cdots$ are Hermitian. Then to treat the general case, we should apply the special result to the above-mentioned Hermitian "real and imaginary parts" and by taking the linear combination we should obtain the desired result since the notion of "restricted metrical convergence" holds for linear combinations of series if it holds for the series themselves. Thus we may assume in what follows that $A, A_1, A_2, \cdots$ are Hermitian operators.

Now form the function

$$\psi(\lambda)\begin{cases} = \lambda & \text{for } |\lambda| \leqq |||A||| \\ = \dfrac{|||A|||^2}{\lambda} & \text{for } |\lambda| \geqq |||A|||. \end{cases}$$

Lemma 1.5.3 applies to $\psi(\lambda)$ yielding $\omega_2(\epsilon)$.

Consider a $\nu = 1, 2, \cdots$. Choose a $\nu' = 1, 2, \cdots$ (depending on ν as well as on $A, A_1, A_2, \cdots$) with

$$[[A_{\nu'} - A]] < \omega_2\left(\frac{1}{\nu}\right).$$

Then Lemma 1.5.3 gives $[[\psi(A_{\nu'}) - \psi(A)]] < 1/\nu$. Since $\psi(\lambda) = \lambda$ for all λ with $|\lambda| \leqq |||A|||$, $\psi(A) = A$.[9] Thus we have

$$[[\psi(A_{\nu'}) - A]] < 1/\nu. \tag{1.5.β}$$

Form next the function

$$\theta(\lambda)\begin{cases} = 1 & \text{for } |\lambda| \leqq |||A||| \\ = |||A|||^2/\lambda^2 & \text{for } |\lambda| \geqq |||A|||. \end{cases}$$

It is clear that $\psi(\lambda) = \lambda \cdot \theta(\lambda)$. Choose a polynomial $p_\nu(\lambda)$ with

$$\theta(\lambda) \geqq p_\nu(\lambda) \geqq \text{Max } (\theta(\lambda) - 1/\nu \cdot |\lambda|, 0) \quad \text{for } |\lambda| \leqq |||A_{\nu'}|||.^{10}$$

[8] Since $[[A]] = [[A^*]]$, $A \to A^*$ is an isometric mapping.

[9] The set, $|\lambda| \leqq |||A|||$, contains the entire spectrum of A.

[10] This is an easy variant of the Weierstrass (polynomial) approximation theorem.

Then the polynomial $q_\nu(\lambda) = \lambda p_\nu(\lambda)$ has no constant term and we have

$$| q_\nu(\lambda) - \psi(\lambda) | \leqq 1/\nu \quad \text{for} \quad | \lambda | \leqq ||| A_{\nu'} ||| .$$

Furthermore $0 \leqq q_\nu(\lambda) \leqq \psi(\lambda) \leqq ||| A |||$ and thus

$$| q_\nu(\lambda) | \leqq ||| A ||| \quad \text{for} \quad | \lambda | \leqq ||| A_{\nu'} |||.$$

These two inequalities imply

$$||| q_\nu(A_{\nu'}) - \psi(A_{\nu'}) ||| \leqq 1/\nu^{11}$$

and hence, *a fortiori*

$$(1.5.\gamma) \qquad [[q_\nu(A_{\nu'}) - \psi(A_{\nu'})]] \leqq 1/\nu$$

and

$$(1.5.\delta) \qquad ||| q_\nu(A_{\nu'}) ||| \leqq ||| A |||.^{11}$$

Combining (1.5.β) and (1.5.γ) gives

$$(1.5.\epsilon) \qquad [[q_\nu(A_{\nu'}) - A]] < 2/\nu.$$

Now put $A'_\nu = q(A_{\nu'})$. Then $A_{\nu'} \epsilon \mathbf{S}$ implies $q_\nu(A_{\nu'}) \epsilon \mathbf{S}$ since $q_\nu(\lambda)$ has no constant term. Hence the sequence $A'_1, A'_2, \cdots$ consists of elements of $\mathbf{S}$, and it is restrictedly metrically convergent to A by (1.5.δ) and (1.5.ϵ).

Observe that in the above proof we even had $||| A_{\nu'} ||| \leqq ||| A |||$ when A is Hermitian. This could be extended to all A but we shall not pursue this matter further.

§1.6 The desired results are now immediate.

LEMMA 1.6.1. *Restricted metric closure of an algebra* $\mathbf{S}$ *implies its metric closure.*

PROOF. Immediate by Lemma 1.5.4.

THEOREM I. *For an algebra* $\mathbf{S}$ *within a finite* $\mathbf{M}$ *the following eight notions of closure are equivalent to each other*:

α) *Restricted metric closure*
β) *Metric closure*
γ) *Weak (topological) closure*
δ) *Weak convergence closure*
ϵ) *Strong (topological) closure*
θ) *Strong convergence closure*
ζ) *Strongest (topological) closure*
η) *Strongest convergence closure.*

PROOF. The implications $\epsilon) \rightarrow \alpha)$, $\beta) \rightarrow \epsilon)$, $\alpha) \rightarrow \beta)$ were established in Lemmas 1.4.1, 1.4.2, 1.6.1, respectively. Consequently $\alpha) \rightleftarrows \beta) \rightleftarrows \epsilon)$. Also

[11] The set, $| \lambda | \leqq ||| A_{\nu'} |||$, contains the entire spectrum of $A_{\nu'}$.

Lemma 1.3.2 yields $\alpha) \rightleftarrows \theta)$. From [2], p. 396, end of §3, we have $\gamma) \rightleftarrows \epsilon)$. [4], p. 112, beginning of §3, yields $\gamma) \rightleftarrows \zeta)$. Also [4], p. 112, end of §2, implies $\theta) \rightleftarrows \eta)$. These equivalences give together

$$\alpha) \rightleftarrows \beta) \rightleftarrows \gamma) \rightleftarrows \epsilon) \rightleftarrows \theta) \rightleftarrows \zeta) \rightleftarrows \eta).$$

And obviously

$$\gamma) \rightarrow \delta) \rightarrow \theta).$$

Thus $\delta)$ also is equivalent to the others.

Thus all those topologizations of the space of all operators, which may be reasonably used in this connection, turn out to be equivalent to each other for the algebras **S** in a factor **M** which is in a finite case.[4]

We also state explicitly

THEOREM II. *The notion of a subring* **S** *of* **M**, *for a finite* **M**, *is purely algebraical.*

PROOF. The equivalences of Theorem I now permit us to apply the argument given at the end of §1.2.

Thus the desideratum expressed at the end of §1.1 is fully realized.

CHAPTER II. THE ALGEBRAICAL TYPE AND ITS OPERATIONS.

THE FUNDAMENTAL GROUP

§2.1 We shall call the abstraction of a ring **M** with respect to algebraical isomorphism, its algebraical type, i.e. two rings $\mathbf{M}_1$ and $\mathbf{M}_2$ will be said to have the same algebraical type if and only if they are algebraically isomorphic. We denote the algebraic type of the ring **M** by $\overline{\mathbf{M}}$. Notions in **M** which are purely algebraical, i.e. invariant under algebraical isomorphisms, will be said to be notions concerning the algebraical type $\overline{\mathbf{M}}$ (and not **M** itself).[12]

Thus the properties enumerated in §1.1 (or equally in [9]) are properties of $\overline{\mathbf{M}}$ for every ring **M**. The notion of a subring of **M** concerns $\overline{\mathbf{M}}$ when **M** is a factor in a finite class (cf. Theorem II). This is also true of $D_{\mathbf{M}}(E)$, $Tr_{\mathbf{M}}(A)$, $[[A]]$ (cf. the beginning of §1.2), and the various notions of closure, as enumerated in Theorem I, when they concern an algebra **S**. Thus when **M** is in a finite class, the properties concerning $\overline{\mathbf{M}}$ are particularly extensive.

§2.2 As in the introduction, §2, let for each $i = 1, 2$ a Hilbert space $\mathfrak{H}_i$ and an operator ring $\mathbf{M}_i$ in $\mathfrak{H}_i$ be given. We generalize $B)$, loc. cit., as follows:

$B^*)$ General (algebraic ring) isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$. This is a one-to-one mapping of $\mathbf{M}_1$ on $\mathbf{M}_2$ which leaves the entity 1 and the operations A^*, $A + B$ invariant; while it carries the entity αA (α any complex number) into either αA always (case a') or into $\bar{\alpha} A$ always (case b') and AB into AB always

[12] This definition should be compared and contrasted with that of spatial type in §3.3.

(case a'') or into BA always (case b''). More specifically, we talk according to which combination of the above cases occurs, of an $[a', a'']$ (or "proper") $[b', a'']$ (or "conjugate") $[a'b'']$ (or "dual") or $[b', b'']$ (or "conjugate dual") isomorphism.

The existence of a general isomorphism, of any one of the above four kinds, between $\mathbf{M}_1$ and $\mathbf{M}_2$ is clearly a property of the types $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$. If a (proper) isomorphism holds, we have $\overline{\mathbf{M}}_1 = \overline{\mathbf{M}}_2$. If a conjugate or a dual isomorphism holds, then $\overline{\mathbf{M}}_1$ determines $\overline{\mathbf{M}}_2$ uniquely, assuming its existence. Hence we may express these relationships by $\overline{\mathbf{M}}_1^{\backprime} = \overline{\mathbf{M}}_2$ or $\overline{\mathbf{M}}_1' = \overline{\mathbf{M}}_2$ respectively.

We must now deduce the existential and other properties of the operations $\overline{\mathbf{M}}^{\backprime}$ and $\overline{\mathbf{M}}'$.

§2.3 Let a Hilbert space $\mathfrak{H}$ be given. Modify the fundamental operations $\alpha f, f + g, (f, g)$ in $\mathfrak{H}$ by replacing them by $\bar{\alpha} f, f + g, \overline{(f, g)}$. Denote the set $\mathfrak{H}$ with the new definition of its fundamental operations by $\mathfrak{H}_c$. Clearly $\mathfrak{H}_c$ is also a Hilbert space; every (linear bounded) operator A in $\mathfrak{H}$ is also one in $\mathfrak{H}_c$, and every ring of operators $\mathbf{M}$ in $\mathfrak{H}$ is also one in $\mathfrak{H}_c$. But we shall denote the A, $\mathbf{M}$ of $\mathfrak{H}$, when considered in $\mathfrak{H}_c$, by A_c, $\mathbf{M}_c$.

Now consider an operator ring $\mathbf{M}$ in $\mathfrak{H}$. The identical mapping $\mathfrak{J}^\circ$ then maps $\mathbf{M}$ in $\mathfrak{H}$ on $\mathbf{M}_c$ in $\mathfrak{H}_c$ and it is in this aspect a conjugate isomorphism of $\mathbf{M}$ and $\mathbf{M}_c$.

Consider next the mapping $\mathfrak{J}^*$, $A \to A^*$. This maps $\mathbf{M}$ in $\mathfrak{H}$ on $\mathbf{M}$ in $\mathfrak{H}$ and it is, in this aspect, a conjugate dual isomorphism of $\mathbf{M}$ and $\mathbf{M}$. Consequently $\mathfrak{J}^*\mathfrak{J}^\circ$ (i.e. $\mathfrak{J}^*$ in a new aspect) is a dual isomorphism of $\mathbf{M}$ and $\mathbf{M}_c$.

Thus we have

THEOREM III. *$\overline{\mathbf{M}}^{\backprime}$ and $\overline{\mathbf{M}}'$ always exist and are single-valued. They are both equal to the above $\overline{\mathbf{M}}_c$. We shall denote them by $\overline{\mathbf{M}}^c$.*

Clearly

$$\overline{\mathbf{M}}^{cc} = \overline{\mathbf{M}} \tag{2.3.α}$$

since $\mathfrak{H}_{c,c} = \mathfrak{H}$, $A_{c,c} = A$, $\mathbf{M}_{c,c} = \mathbf{M}$.

Observe that the definitions of a factor and of the case to which it belongs are unaffected by conjugate isomorphisms. For these definitions make use of αA only in defining the set of all $\alpha 1$ but never for a specific numerical value of α. To see this, reconsider [5], p. 138, Def. 3.1.2; p. 173, Theorem IX. Hence the operation $\overline{\mathbf{M}}^c$ (viewed as $\overline{\mathbf{M}}^{\backprime}$) does not affect the factor character of $\mathbf{M}$ nor the case to which it belongs.

§2.4 In what follows we shall consider matrices of operators. Given a $p = 1, 2, \cdots, \infty$ a *p-order matrix* $\langle A_{t,s} \rangle$ is a scheme or array of operators $A_{t,s}$ in which the indices s and t range from 1 to p for a finite p and over $1, 2, \cdots$ if $p = \infty$. (This last will be abbreviated $s, t \leqq p$ in both cases.) An $\langle A_{t,s} \rangle$ is finite if there exists a finite $q = 1, 2, \cdots$ such that $A_{t,s} = 0$ if either $t > q$ or $s > q$. For a finite p this is automatically fulfilled, but it is a real restriction for $p = \infty$.

Now as in the Introduction, §2, let, for each $i = 1, 2$, a Hilbert space $\mathfrak{H}_i$

and an operator ring $\mathbf{M}_i$ in $\mathfrak{H}_i$ be given. We generalize B), loc. cit., as follows:

B_p) *p-matrix (algebraic ring) isomorphism of* $\mathbf{M}_1$ *over* $\mathbf{M}_2$. This is a one-to-one mapping of $\mathbf{M}_1$ on a set $\mathfrak{J}$ of p-order matrices $\langle A_{t,s}\rangle$ over $\mathbf{M}_2$ (i.e. $A_{t,s} \in \mathbf{M}_2$) which is algebraically ring-isomorphic in the matrix sense, i.e. it carries 1, αA°, $A^{\circ *}$ $A^\circ + B^\circ$, $A^\circ B^\circ$ in $\mathbf{M}_1$ into $\langle \delta_{t,s} 1\rangle$, $\langle \alpha A_{t,s}\rangle$ $\langle A^*_{s,t}\rangle$, $\langle A_{t,s} + B_{t,s}\rangle$, $\langle \sum_{r=1}^{p} A_{r,s} B_{t,r}\rangle$ in $\mathfrak{J}$. The $\sum_{r=1}^{p}$ in the above expression, when $p = \infty$ must converge in the sense of the strong operator topology, irrespective of the order of summation.

The set $\mathfrak{J}$ must, at any rate, contain all finite matrices $\langle A_{t,s}\rangle$ over $\mathbf{M}_2$. (Cf. [5], pp. 136–137, Lemma 2.4.3, where the same notions are used, the same multiplication convention obtains, as well as the same notion of convergence for $\sum_{r=1}^{\infty}$.)

For a given p, $\mathbf{M}_1$, $\mathbf{M}_2$ the existence of such a p-matrix isomorphism of $\mathbf{M}_1$ over $\mathbf{M}_2$ is a property of types $\overline{\mathbf{M}}_1$, $\overline{\mathbf{M}}_2$. This is obvious for $p = 1, 2, \cdots$ while for $p = \infty$ it is only necessary to remember the purely algebraical character of the notion of strong convergence (cf. the beginning of §1.1, or [9], Theorem II). Whether $\overline{\mathbf{M}}_1$ (assuming its existence) is uniquely determined by $\overline{\mathbf{M}}_2$ or not, for a given $p = 1, 2, \cdots, \infty$ depends obviously on whether the set $\mathfrak{J}$ in B_p) is uniquely determined or not. For $p = 1, 2, \cdots$ (p finite), the last part of B_p) implies that the set $\mathfrak{J}$ must be the set of all $\langle A_{t,s}\rangle$ with $A_{t,s} \in \mathbf{M}_2$ since they are all finite. Hence in this case $\overline{\mathbf{M}}_1$ is uniquely determined. For $p = \infty$ the question of uniqueness is not so immediately settled. We shall prove that $\mathfrak{J}$ and with it $\overline{\mathbf{M}}_1$ are again unique, but we delay this discussion to the next section. Nevertheless, for $p = 1, 2, \cdots, \infty$ we express the existence of a p-order matrix isomorphism of $\mathbf{M}_1$ over $\mathbf{M}_2$ as a relationship between $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$ by $\overline{\mathbf{M}}_1 = \overline{\mathbf{M}}_2^p$. But then, quite apart from the question of existence, we may assume that $\overline{\mathbf{M}}^p$ is single-valued only for a finite p. At present we must consider the possibility that for $p = \infty$, $\overline{\mathbf{M}}^p$ is many valued.

The general existence of $\overline{\mathbf{M}}^p$ belongs in abstract algebra, but we shall need $\overline{\mathbf{M}}^p$ as an operator ring. For this reason, as well as for the sake of general orientation, we shall construct it explicitly within the framework of [5], pp. 135–137, §2.4.

Let a Hilbert space $\mathfrak{H}$ and a ring $\mathbf{M}$ in it be given, and a $p = 1, 2, \cdots, \infty$. We perform the constructions of [5], loc. cit., the $\mathfrak{H}_2$ there being our $\mathfrak{H}$ and the $\mathfrak{H}_1$ there, any p-dimensional unitary space. (This is not the $\mathfrak{H}_1$ of B_p) above. We conform to the notations of [5], loc. cit.) Form $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and the ring $\mathbf{B}\otimes$ of all its bounded operators. Then all elements of $\mathbf{B}\otimes$ can be represented as p order matrices $\langle A_{t,s}\rangle$ ($t, s \leqq p$, $A_{t,s}$ in $\mathfrak{H}_2 = \mathfrak{H}$). (Cf. [5], p. 136, Def. 2.4.2 and Lemma 2.4.2.) For the given ring $\mathbf{M}$, form the set $\mathbf{M}_1$ consisting of all operators in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ which have a matrix $\langle A_{t,s}\rangle$ with all $A_{t,s} \in \mathbf{M}$. $\mathfrak{J}$ is the set of all those $\langle A_{t,s}\rangle$ with $A_{t,s} \in \mathbf{M}$ which correspond to some operator in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. From this it follows immediately that $\mathfrak{J}$ contains all finite matrices $\langle A_{t,s}\rangle$ with $A_{t,s} \in \mathbf{M}$.[13] Now $\mathbf{M}_1$ in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $\mathbf{M}$ in $\mathfrak{H} = \mathfrak{H}_2$ together with

[13] For a finite matrix $\langle A_{t,s}\rangle$ an operator A° in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ with $A^\circ \frown \langle A_{t,s}\rangle$ can be constructed in the sense of [5], p. 136, Def. 2.4.2, without any convergence difficulties.

the interpretation of the matrix scheme $\langle A_{t,s}\rangle$ which we are now using, clearly fulfill all the requirements of B_p). (We have our $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathfrak{H} = \mathfrak{H}_2$, $\mathbf{M}_1$, $\mathbf{M}$ in place of the $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathbf{M}_1$, $\mathbf{M}_2$ of B_p).)

Summing up the results obtained so far, we have

THEOREM IV. *$\bar{\mathbf{M}}^p$ exists for all $p = 1, 2, \cdots, \infty$ and it is certainly one-valued for $p = 1, 2, \cdots$ (p finite).*

Our definitions in B_p) and that of §2.3 give together

$$(\bar{\mathbf{M}}^c)^p = (\bar{\mathbf{M}}^p)^c, \qquad (p = 1, 2, \cdots), \tag{2.4.α}$$

and B_p) gives

$$(\bar{\mathbf{M}}^p)^q = \bar{\mathbf{M}}^{pq} \qquad (p, q = 1, 2, \cdots). \tag{2.4.β}$$

(The exponent ∞ will be considered in the next section.)

§2.5 In this section we shall make use of the operations $A_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$ defined in [1], p. 78, and [5], p. 186, Def. 11.3.1. If A is an operator in $\mathfrak{H}$ which is reduced by the closed linear set $\mathfrak{M}$ then $A_{(\mathfrak{M})}$ is its part in $\mathfrak{M}$. And $\mathbf{M}_{(\mathfrak{M})}$ is the set of all $A_{(\mathfrak{M})}$ for all $A \in \mathbf{M}$ which are reduced by $\mathfrak{M}$. $A_{(\mathfrak{M})}$, $\mathbf{M}_{(\mathfrak{M})}$ are in $\mathfrak{M}$ while A, $\mathbf{M}$ were in $\mathfrak{H}$.

LEMMA 2.5.1. *Assume $E = P_{\mathfrak{M}} \in \mathbf{M}$. Let $\mathbf{M}^E$ be the set of all $A \in \mathbf{M}$ for which $EA = AE = A$. Then we have*

(i) *$A_{(\mathfrak{M})}$ can be formed for all $A \in \mathbf{M}^E$.*

(ii) *$\mathbf{M}_{(\mathfrak{M})}$ coincides with $(\mathbf{M}^E)_{(\mathfrak{M})}$.*

(iii) *$A \to A_{(\mathfrak{M})}$ is a one-to-one mapping of $\mathbf{M}^E$ on $\mathbf{M}_{(\mathfrak{M})}$ which carries E (in $\mathbf{M}^E$, i.e. in $\mathfrak{H}$) into 1 (in $\mathbf{M}_{(\mathfrak{M})}$ i.e. in $\mathfrak{M}$) and leaves the operations αA, A^*, $A + B$, AB invariant.*

PROOF. Ad (i) Every $A \in \mathbf{M}^E$ commutes with $E = P_{(\mathfrak{M})}$, i.e. it is reduced by $\mathfrak{M}$.

Ad (ii) $A \in \mathbf{M}$ implies $EAE \in \mathbf{M}$ and thence clearly $EAE \in \mathbf{M}^E$. Now if $\mathfrak{M}$ reduces A, then both A and EAE have the same part in $\mathfrak{M}$, i.e. $A_{(\mathfrak{M})} = (EAE)_{(\mathfrak{M})}$. Thus $(\mathbf{M}^E)_{(\mathfrak{M})} \supseteq \mathbf{M}_{(\mathfrak{M})}$. But $\mathbf{M}^E \subseteq \mathbf{M}$ hence also $(\mathbf{M}^E)_{(\mathfrak{M})} \subset \mathbf{M}_{(\mathfrak{M})}$. So $(\mathbf{M}^E)_{(\mathfrak{M})} = \mathbf{M}_{(\mathfrak{M})}$.

Ad (iii) Every $A \in \mathbf{M}^E$ is reduced by $\mathfrak{M}$ (cf. (i) above) and its part in $\mathfrak{M}^\perp$ ($= \mathfrak{H} \ominus \mathfrak{M}$) is clearly 0. So we may view it as an operator in $\mathfrak{M}$ as well as in $\mathfrak{H}$, and in its former aspect it is $A_{(\mathfrak{M})}$. All our assertions are now immediate.

Consider now $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathfrak{H}_1$, $\mathfrak{H}_2$ and an ∞-matrix isomorphism of $\mathbf{M}_1$ over $\mathbf{M}_2$ as in B_∞) in the last section. For any $q = 1, 2, \cdots$ form the set $\mathfrak{J}^q$ of all matrices $\langle A_{t,s}\rangle$ for which all $A_{t,s} \in \mathbf{M}_2$ and where $A_{t,s} = 0$ when $t > q$ or $s > q$. Form also the matrix $\langle e^q_{t,s} 1\rangle$ where $e^q_{t,s} = 1$ if $t = s \leqq q$ and $e^q_{t,s} = 0$ otherwise. Then it is easy to establish that

LEMMA 2.5.2. (i) $\mathfrak{J}^q \subseteq \mathfrak{J}$.

(ii) *$\langle e^q_{t,s}\rangle \in \mathfrak{J}^q$ corresponds to a projection $E^q \in \mathbf{M}$. Put $E^q = P_{\mathfrak{M}^q}$.*

(iii) *$E^1 \leqq E^2 \leqq \cdots$ and $\lim_{q\to\infty} E^q = 1$ in the strong sense.*

(iv) *Every matrix $\langle A_{t,s}\rangle \in \mathfrak{J}^q$ can be viewed both as a q^{th} order matrix and as an ∞-order matrix. It corresponds in its former aspect to an element of a ring*

$\mathbf{M}_2^q$ *with* $\overline{\mathbf{M}_2^q} = \overline{\mathbf{M}}_2^q$ *and in its latter aspect to an element of* $(\mathbf{M}_1)^{E^q} \subseteq \mathbf{M}_1$. *The latter correspondence can be continued by Lemma* 2.5.1 *to one with an element of* $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$. *These elements exhaust the sets* $\mathbf{M}_2^q$, $\mathbf{M}_1^{E^q}$ *and* $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$, *respectively.*

(v) *The correspondence of* (*iv*) *is an algebraic ring isomorphism of* $\mathbf{M}_2^q$ *and* $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$. *Hence* $\overline{(\mathbf{M}_1)_{(\mathfrak{M}^q)}} = \overline{\mathbf{M}_2^q} = \overline{\mathbf{M}}_2^q$.

PROOF. Ad (i). Immediate by the final clause of B_∞).

Ad (ii). Clearly $\langle e_{t,s}^q \cdot 1 \rangle \in \mathfrak{J}^q$ hence it is $\in \mathfrak{J}$ and thus corresponds to an element of $\mathbf{M}_1$. The rules of matrix computation show that this element is a projection.

Ad (iii). The rules of matrix computation show that $E^1 \leqq E^2 \leqq \cdots$. Now if $f = \langle f_1, f_2, \cdots \rangle \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$ then $E^q f = \langle f_1, \cdots, f_q, 0, \cdots \rangle \to f$ as $q \to \infty$. Hence $\lim_{q\to\infty} E^q$ is .

Ad (iv). The $\langle A_{t,s} \rangle \in \mathfrak{J}^q$ are clearly characterized by $\langle e_{t,s}^q \cdot 1 \rangle \cdot \langle A_{t,s} \rangle = \langle A_{t,s} \rangle \langle e_{t,s}^q \cdot 1 \rangle = \langle A_{t,s} \rangle$ and thus we may add by (i), $\langle A_{t,s} \rangle \in \mathfrak{J}$. Hence the second correspondence mentioned maps the $\langle A_{t,s} \rangle \in \mathfrak{J}^q$ precisely on the $A^\circ \in \mathbf{M}$ with $E^q A^\circ = A^\circ E^q = A^\circ$ i.e. on the $A^\circ \in (\mathbf{M}_1)^{E^q}$. All other assertions are obvious, remembering in particular Lemma 2.5.1.

Ad (v). The corresponding assertions for the relationship between $\mathbf{M}_2^q$ and $(\mathbf{M}_1)^{E^q}$ follow immediately from the rules of matrix computation. They are transferred to $\mathbf{M}_2^q$ and $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$ by using Lemma 2.5.1.

We now go further:

LEMMA 2.5.3. *A matrix* $\langle A_{t,s} \rangle \in \mathfrak{J}^q$ *corresponds to three operators in the sense of* (*iv*) *in Lemma* 2.5.2, *i.e. to an operator of* $\mathbf{M}_2^q$ *to one in* $(\mathbf{M}_1)^{E^q}$ *and to one in* $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$. *All three operators have the same bound,* $|||\ \cdots\ |||$ *and we denote this common bound by* $|||\ \langle A_{t,s} \rangle\ |||$.

PROOF. The operator in $(\mathbf{M}_1)^{E^q}$ and that one in $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$ have the same bound, since we are considering the same operator, once in $\mathfrak{H}_1$ and once in $\mathfrak{M}^q$.[14] The operator in $\mathbf{M}_2^q$ and that one in $(\mathbf{M}_1)_{(\mathfrak{M}^q)}$ have the same bound since they obtain from one another by an algebraical ring isomorphism (cf. (v) in Lemma 2.5.2) and since the numerical value of the bound is a purely algebraical notion (cf. §1.1.). This completes the proof.

LEMMA 2.5.4. *Consider a matrix* $\langle A_{t,s} \rangle$ *all* $A_{t,s} \in \mathbf{M}_2$. *Define the matrices* $\langle A_{t,s}^q \rangle \in \mathfrak{J}^q$ *by the equations* $A_{t,s}^q = A_{t,s}$, *if* $t \leqq q$ *and* $s \leqq q$ *and* $A_{t,s}^q = 0$ *otherwise. Then* $\langle A_{t,s} \rangle \in \mathfrak{J}$ *if and only if the numerical sequence* $|||\ \langle A_{t,s}^1 \rangle\ |||$, $|||\ \langle A_{t,s}^2 \rangle\ |||$, $\cdots$ (*cf. Lemma* 2.5.3 *above*) *is bounded.*

PROOF. Necessity: Assume $\langle A_{t,s} \rangle \in \mathfrak{J}$ and let it correspond to $A^\circ \in \mathbf{M}_1$. Then, clearly $\langle A_{t,s}^q \rangle = \langle e_{t,s}^q \cdot 1 \rangle \cdot \langle A_{t,s} \rangle \langle e_{t,s}^q \cdot 1 \rangle$ i.e. $\langle A_{t,s}^q \rangle$ corresponds to the operator $E^q A^\circ E^q \in \mathbf{M}_1$. Clearly $E^q A^\circ E^q \in (\mathbf{M}_1)^{E^q}$. Thus the second definition of Lemma 2.5.3 gives $|||\ \langle A_{t,s}^q \rangle\ ||| = |||\ E^q A^\circ E^q\ |||$. Now we have in $\mathfrak{H}_1$ that $|||\ E^q A^\circ E^q\ ||| \leqq |||\ E^q\ ||| \cdot |||\ A^\circ\ ||| \cdot |||\ E^q\ ||| = |||\ A^\circ\ |||$. So $|||\ \langle A_{t,s}^q \rangle\ ||| \leqq |||\ A^\circ\ |||$. This holds for $q = 1, 2, \cdots$ and hence the sequence $|||\ \langle A_{t,s}^1 \rangle\ |||$, $|||\ \langle A_{t,s}^2 \rangle\ |||$, $\cdots$ is bounded.

Sufficiency: Assume that the numerical sequence $|||\ \langle A_{t,s}^1 \rangle\ |||$, $|||\ \langle A_{t,s}^2 \rangle\ |||$, $\cdots$

[14] The part of the first-mentioned operator in $(\mathfrak{M}^q)^-$ is 0.

is bounded. $\langle A^q_{t,s}\rangle \in \mathfrak{J}^q$ so it corresponds to an $A^q \in (\mathbf{M}_1)^{E^q} \subseteq \mathbf{M}_1$. Owing to the definition of Lemma 2.5.3, $|||\langle A^q_{t,s}\rangle||| = |||A^q|||$ hence

(2.5.α) The (numerical) sequence $|||A^1|||, |||A^2|||, \cdots$ is bounded.

For $q \geqq r$ clearly $\langle e^r_{t,s}\cdot 1\rangle\langle A^q_{t,s}\rangle\langle e^r_{t,s}\cdot 1\rangle = \langle A^r_{t,s}\rangle$ hence $E^rA^qE^r = A^r$.

Now consider an $f \in \mathfrak{M}^r$.

For $q \geqq r$, let $f_q = A^qf$. Since $||f_q||^2 \leqq |||A^q|||^2 \cdot ||f||^2$, these f_q's are bounded. Also $f \in \mathfrak{M}^r \subset \mathfrak{M}^q$ so $E^qf = f$. Hence for $q' \geqq q \geqq r$, $E^qf_{q'} = E^qA^{q'}f = E^qA^{q'}E^qf = A^qf = f_q$. So $f_{q'} - f_q$ and f_q are orthogonal, $||f_{q'} - f_q||^2 + ||f_q||^2 = ||f_{q'}||^2$ i.e.

$$||f_{q'} - f_q||^2 = ||f_{q'}||^2 - ||f_q||^2 \qquad \text{for } q' \geqq q \geqq r \tag{2.5.β}$$

Thus the $||f_r||^2, ||f_{r+1}||^2, \cdots$ form a monotone bounded sequence, and hence the $\lim_{q\to\infty}||f_q||^2$ exists. Therefore the right-hand side of (2.5.β) converges to zero for $q, q' \to \infty$. Hence (2.5.β) now implies that strong limit A^qf exists for $q \to \infty$. The restriction $q \geqq r$ may now be dropped.

The set of all f for which A^qf is strongly convergent is closed, since the $|||A^q|||$ are uniformly bounded. This set is obviously linear and by the above it contains $\mathfrak{M}^1, \mathfrak{M}^2, \cdots$. Thus we see

(2.5.γ) The sequence $A^1, A^2, \cdots$ converges for all f in the closed linear set determined by $\mathfrak{M}^1, \mathfrak{M}^2, \cdots$.

Since the $\mathfrak{M}^1, \mathfrak{M}^2, \cdots$ are together dense, the set of (2.5.γ) is necessarily $\mathfrak{H}$ itself. Denote the limit of $A^1f, A^2f, \cdots$ by Af. Then A belongs to $\mathbf{M}$ along with $A^1, A^2, \cdots$. Thus we have shown

(2.5.δ) The sequence $A^1, A^2, \cdots$ converges for all f in $\infty \otimes \mathfrak{H}$ to an $A \in \mathbf{M}$ in the strong sense.

Let A correspond to the matrix $\langle B_{t,s}\rangle$ in B_∞). Of course $\langle B_{t,s}\rangle \in \mathfrak{J}$. Since for $q \geqq r$, $E^rA^qE^r = A^r$, (2.5.δ) implies that $E^rAE^r = A^r$. Hence $\langle e^r_{t,s}\cdot 1\rangle\langle B_{t,s}\rangle\langle e^r_{t,s}\cdot 1\rangle = \langle A^r_{t,s}\rangle$. Thus the rules of matrix computation yield $B^r_{t,s} = A^r_{t,s}$. Now let t, s be given and choose $r = \text{Max}(t, s)$. Then the above formula gives $B_{t,s} = A_{t,s}$. Since t, s were arbitrary, this means $\langle B_{t,s}\rangle = \langle A_{t,s}\rangle$ and consequently $\langle A_{t,s}\rangle \in \mathfrak{J}$ too.

Thus the proof is completed.

LEMMA 2.5.5. *The set $\mathfrak{J}$ can be uniquely and purely algebraically characterized in terms of $\mathbf{M}_2$ alone.*

PROOF. A unique characterization of $\mathfrak{J}$ was given in Lemma 2.5.4. By using the first definition of Lemma 2.5.3 for $|||\langle A^q_{t,s}\rangle|||$ that one in terms of $\mathbf{M}^q_2$ it becomes clear that this characterization is purely algebraical in terms of $\mathbf{M}_2$ and of the $\mathbf{M}^q_2$, $q = 1, 2, \cdots$. But we have already reduced the $\mathbf{M}^q_2$ i.e. the $\mathbf{M}^q_2 = \mathbf{M}^q_2$ for q finite, to $\mathbf{M}_2$ by Theorem IV.

The proof is thus completed.

The remarks made in §2.4, preceding Theorem IV, in conjunction with the above Lemma 2.5.5, permit us now to assert

THEOREM IV′. $\overline{\mathbf{M}}^{\infty}$ *too is one-valued.*

Our definitions in B_∞ and that of §2.3 give together

$$(\overline{\mathbf{M}}^{\epsilon})^{\infty} = (\overline{\mathbf{M}}^{\infty})^{\epsilon}. \tag{2.5.ϵ}$$

(This corresponds to (2.4.α). We could extend (2.4.β) too, but we shall not need it here. Furthermore this extension is an easy consequence of Lemma 3.1.6.)

§2.6 We undertake now to find the inverse operation to $\mathbf{N}^p$. At first we restrict neither the ring $\mathbf{N}$ nor the exponent $p = 1, 2, \cdots, \infty$.

DEFINITION 2.6.1. *A system of* p^2 *operators* $W_{\mu,\nu}$ $(\mu, \nu \leqq p)$ *is a system of p-matrix units if it possesses the following properties*:

(i) $W^*_{\mu,\nu} = W_{\nu,\mu}$

(ii) $W_{\mu,\nu}W_{\sigma,\omega} = W_{\mu,\omega}$ *for* $\nu = \sigma$

$= 0$ *if* $\nu \neq \sigma$.

Some immediate properties of these systems are given in

LEMMA 2.6.1. *Every system of p-matrix units* $W_{\mu,\nu}$ $(\mu, \nu \leqq p)$ *possesses the following properties*:

(i) *The* $W_{\mu,\mu}$ $(\mu \leqq p)$ *are pairwise orthogonal projections.*

(ii) *Every* $W_{\mu,\nu}$ *is partially isometric with the initial projection* $W_{\nu,\nu}$ *and the final projection* $W_{\mu,\mu}$ (cf. [5], §4.3).

(iii) *Form* $E_0 = \sum_{\mu=1}^{p} W_{\mu,\mu}$. *This* $\sum_{\mu=1}^{p}$ *is either a finite sum or when* $p = \infty$ *converges in the sense of the strong operator topology, irrespective of the order in which the* $\mu = 1, 2, \cdots$ *are gone through. This* E_0 *is a projection and it is the unit of the system* $W_{\mu,\nu}$ *i.e. always*

$$E_0 W_{\mu,\nu} = W_{\mu,\nu} E_0 = W_{\mu,\nu}.$$

PROOF. Ad (i), (iii): The convergence in (iii) is a consequence of (i), as is seen from familiar considerations concerning pairwise orthogonal projections (cf. e.g. [5], pp. 76–78, or the proof of (2.5.δ) in the proof of Lemma 2.5.4 above). All other assertions are immediately verified by using (i), (ii) in Def. 2.6.1.

Ad (ii). Definition 2.6.1 gives directly

$$W^*_{\mu,\nu}W_{\mu,\nu} = W_{\nu,\nu}, \qquad W_{\mu,\nu}W^*_{\mu,\nu} = W_{\mu,\mu}.$$

Since $W_{\nu,\nu}$, $W_{\mu,\mu}$ are projections by (i) above, these formulae imply our assertions by [5], p. 142, Lemma 4.3.2 (the last two criteria), and p. 141, Lemma 4.3.1.

We prove now

THEOREM V. $\overline{\mathbf{M}} = \overline{\mathbf{N}}^p$ *is equivalent to this*: *There exists a system of p-matrix units* $W_{\mu,\nu} \epsilon \mathbf{M}$ $(\mu, \nu \leqq p)$ *with the unit* 1 (*cf.* (iii) *in Lemma* 2.6.1) *and with the following further property*:

$W_{1,1}$ is a projection. Let $\mathfrak{M}$ be the closed linear set with $P_{\mathfrak{M}} = W_{1,1}$. Then $\mathbf{M}_{(\mathfrak{M})}$ must be algebraically isomorphic to $\mathbf{N}$.[15]

(Concerning the omission of this extra condition, cf. the corollary below.)

PROOF. Sufficiency: Assume the existence of $W_{\mu,\nu}$ $(\mu, \nu \leqq p)$ as described. $W_{1,1} \epsilon \mathbf{M}$ is a projection. Let $\mathfrak{M}$ be such that $P_{\mathfrak{M}} = E = W_{1,1}$.

For every $A^0 \epsilon \mathbf{M}$ define

$$A'_{t,s} = W_{1,s} A^0 W_{t,1} \qquad (t, s \leqq p). \tag{2.6.α}$$

Clearly $A'_{t,s} \epsilon \mathbf{M}$ and one verifies immediately that $EA'_{t,s} = A'_{t,s}E = A'_{t,s}$ $(E = W_{1,1})$. Hence $A'_{t,s} \epsilon \mathbf{M}^E$. Now apply Lemma 2.5.1 and form

$$A_{t,s} = (A'_{t,s})_{\mathfrak{M}}\,. \tag{2.6.β}$$

Accordingly $A_{t,s} \epsilon \mathbf{M}_{(\mathfrak{M})}$.

Thus to every $A^0 \epsilon \mathbf{M}$ there corresponds a p-order matrix $\langle A_{t,s}\rangle$, $(t, s \leqq p)$ with all $A_{t,s} \epsilon \mathbf{M}_{(\mathfrak{M})}$. Let $\mathfrak{J}$ be the set of these $\langle A_{t,s}\rangle$ which correspond to an $A^0 \epsilon \mathbf{M}$.

We shall now show that this correspondence establishes a p-matrix automorphism of $\mathbf{M}$ over $\mathbf{M}_{(\mathfrak{M})}$ in the sense of $B_{\mathfrak{p}}$) in §2.4. That means $\overline{\mathbf{M}} = \overline{\mathbf{M}}^p_{(\mathfrak{M})}$ and as $\overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{N}}$ so $\overline{\mathbf{M}} = \bar{\mathbf{N}}^p$. This will complete the proof of the sufficiency.

We prove the above assertions in three successive steps.

The correspondence is one-to-one—i.e. if A^0, $B^0 \epsilon \mathbf{M}$ have $\langle A_{t,s}\rangle = \langle B_{t,s}\rangle$ then $A^0 = B^0$. Now $\langle A_{t,s}\rangle = \langle B_{t,s}\rangle$ means $A_{t,s} = B_{t,s}$ for $t, s \leqq p$ and hence $(A'_{t,s})_{(\mathfrak{M})} = (B'_{t,s})_{(\mathfrak{M})}$ and $A'_{t,s} = B'_{t,s}$ since both are $\epsilon \mathbf{M}^E$. Thus $W_{1,s}A^0W_{t,1} = W_{1,s}B^0W_{t,1}$. Multiplication by $W_{s,1}$ on the left and by $W_{1,t}$ on the right, gives $W_{s,s}A^0W_{t,t} = W_{s,s}B^0W_{t,t}$. If we sum over s and t we obtain $A^0 = B^0$ as desired, since by hypothesis $E = 1 = \sum_{s=1}^{p} W_{s,s}$.

The correspondence is a matrix isomorphism. This can be verified by the use of Def. 2.6.1, $E_0 = \sum_{\mu=1}^{p} W_{\mu,\mu} = 1$ and the equations (2.6.α) and (2.6.β). For these show that by the use of Equation (2.6.α) we correspond to 1, αA^0, A^{0*}, $A^0 + B^0$, A^0B^0 (in $\mathbf{M}$) matrices $\langle \delta_{t,s}E\rangle$, $\langle \alpha A'_{t,s}\rangle$, $\langle A'^{*}_{s,t}\rangle$, $\langle A'_{t,s} + B'_{t,s}\rangle$ $\langle \sum_{r=1}^{p} A'_{r,s}B'_{t,r}\rangle$ of elements of $\mathbf{M}^E$ and then by the use of Equation (2.6.β) we correspond these to $\langle \delta_{t,s}1\rangle$, $\langle \alpha A_{t,s}\rangle$, $\langle A^*_{s,t}\rangle$, $\langle A_{t,s} + B_{t,s}\rangle$, $\langle \sum_{r=1}^{p} A_{r,s}B_{t,r}\rangle$ whose elements are in $\mathbf{M}_{(\mathfrak{M})}$.

$\mathfrak{J}$ contains all finite matrices. Consider a finite matrix $\langle A_{t,s}\rangle$ where $A_{t,s} \epsilon \mathbf{M}_{(\mathfrak{M})}$. There is a $q < \infty$ such that $A_{t,s} = 0$ when $t > q$ or when $s > q$. Let $A'_{t,s} \epsilon \mathbf{M}^E$ satisfy (2.6.β) and put

$$A^0 = \sum_{t=1}^{q} \sum_{s=1}^{q} W_{s,1} A'_{t,s} W_{1,t}\,.$$

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[15] Observe that in this presentation $\bar{\mathbf{N}}$ is determined by $\overline{\mathbf{M}}$ and $W_{1,1}$ together. Whether $\overline{\mathbf{M}}$ alone suffices to determine $\bar{\mathbf{N}}$ is a different question. If, however, p is finite and $\mathbf{M}$ is a factor, we do have that $\overline{\mathbf{M}}$ and p determine $\bar{\mathbf{N}}$ uniquely if there is such an $\bar{\mathbf{N}}$. For if $\mathfrak{N}^{(1)}$ is the range of $W_{1,1}$ for one choice of the p-matrix units and $\mathfrak{M}^{(2)}$ that for another, then $D_{\mathbf{M}}(\mathfrak{M}^{(1)}) = (1/p)D_{\mathbf{M}}(\mathfrak{H}) = D_{\mathbf{M}}(\mathfrak{M}^{(2)})$. Hence Lemma 2.8.1 (of our present text) implies, $\mathbf{M}_{(\mathfrak{M}^{(1)})} = \mathbf{M}_{(\mathfrak{M}^{(2)})}$. Hence $\overline{\mathbf{M}_{(\mathfrak{M})}}$ depends only on $\overline{\mathbf{M}}$ and p.

If however p is ∞ then $\mathbf{M}$ itself is in an infinite case by (iii) of Lemma 2.7.1. Under these circumstqnces, $\overline{\mathbf{M}}$ fails to determine $\bar{\mathbf{N}}$ as may be seen from δ) in Lemma 3.1.5 (write $\overline{\mathbf{M}}$ for its $\overline{\mathbf{M}}^{\infty}$).

Then $A^0 \in \mathbf{M}$ and it follows from the rules of Def. 2.6.1 that (2.6.α) holds. Thus $\langle A_{t,s}\rangle$ corresponds to this A^0 and therefore it belongs to $\mathfrak{J}$.

Necessity: Assume $\overline{\mathbf{M}} = \overline{\mathbf{N}}^p$ in the sense of B_p) in §2.4. For every pair $\mu, \nu \leqq p$ form the matrix, $\langle \delta_{t,\nu}\delta_{s,\mu}1\rangle$. This matrix is clearly finite and hence it belongs to $\mathfrak{J}$. Thus it corresponds to an element of $\mathbf{M}$ which we denote by $W_{\mu,\nu}$. Then the matrix computation rules of B_p) give immediately the rules of Def. 2.6.1. Thus the results of Lemma 2.6.1 are now available. Now in the notation of (ii) in Lemma 2.5.2, $E^q = \sum_{\mu=1}^{q} W_{\mu,\mu}$ $(q = 1, 2, \cdots)$. Hence Lemma 2.5.2 (iii) and Lemma 2.6.1 (iii) imply $E_0 = \sum_{\mu=1}^{p} W_{\mu,\mu} = 1$.

Put $P_{\mathfrak{M}} = E = W_{1,1}$. $A^0 \in \mathbf{M}^E$ means $EA^0 = A^0E = A^0$ or, by the above-mentioned rules, that the matrix A^0 has the form $\langle \delta_{t,1}\delta_{s,1}A\rangle$, $A \in \mathbf{N}$. This correspondence between the $A^0 \in \mathbf{M}^E$ and the $A \in \mathbf{N}$ is clearly one-to-one, it carries $E \in \mathbf{M}^E$ (which has the matrix $\langle \delta_{t,1}\delta_{s,1}1\rangle$) into $1 \in \mathbf{N}$ and it leaves the operations αA, A^*, $A + B$, AB invariant. These results, together with Lemma 2.5.1, show therefore that $\mathbf{M}_{(\mathfrak{M})}$ is algebraically isomorphic to $\mathbf{N}$.

Thus all our requirements are fulfilled.

COROLLARY. *Given* $\mathbf{M}$ *and* p, *the equation* $\overline{\mathbf{M}} = \overline{\mathbf{N}}^p$ *possesses a solution* $\overline{\mathbf{N}}$ *if and only if there exists a system of* p*-matrix units,* $W_{\mu,\nu} \in \mathbf{M}$ $(\mu, \nu \leqq p)$ *with the unit* 1.

PROOF. This becomes clear when we omit the last condition in the above theorem, since the only effect of that condition is to determine $\overline{\mathbf{N}}$ in terms of the $W_{\mu,\nu}$.

Under certain conditions we may even go further.

LEMMA 2.6.2. *If* $\mathbf{M}$ *is a factor not in case* (I_n), *and* $p = 2, 3, \cdots$ *is finite, then there is a system of* p*-matrix units* $W_{\mu,\nu}$, $(\mu, \nu \leqq p)$ *with unit* $E_0 = 1$, $W_{\mu,\nu} \in \mathbf{M}$.

PROOF. We first obtain a set $E_1, \cdots, E_p$ of projections in $\mathbf{M}$ such that $D_{\mathbf{M}}(E_i) = D_{\mathbf{M}}(E_1)$ and $\sum_{n=1}^{p} E_n = 1$. Suppose first that $\mathbf{M}$ is in a case (II_1). We may assume that $D_{\mathbf{M}}(1) = 1$. The range of $D_{\mathbf{M}}$ contains $1/p$ (cf. [5], Theorem VIII, p. 172), and hence there is an $E_1 \in \mathbf{M}$ with $D_{\mathbf{M}}(E_1) = 1/p$. Since $D_{\mathbf{M}}(E_1) = 1/p \leqq 1 - 1/p = D_{\mathbf{M}}(1 - E_1)$, there exists a projection $E_2 \leqq 1 - E_1$ with $D_{\mathbf{M}}(E_2) = 1/p = D_{\mathbf{M}}(E_1)$. (Cf. [5], Lemma 8.3.1, p. 167.) Furthermore $D_{\mathbf{M}}(1 - E_1 - E_2) = 1 - 2/p$. If $p > 2$ we can find an $E_3 \leqq 1 - E_1 - E_2$ such that $D_{\mathbf{M}}(E_3) = 1/p$. Thus we may continue until we obtain a set of p pairwise orthogonal projections $E_1, \cdots, E_p \in \mathbf{M}$ with $\sum_{n=1}^{p} E_n = 1$, $D_{\mathbf{M}}(E_n) = 1/p$. If $\mathbf{M}$ is in an infinite case, we can easily modify the proof of [5], Lemma 7.2.3, p. 157, to yield that there are p pairwise orthogonal, dimensionally equivalent, closed, linear sets $\mathfrak{M}_1, \cdots, \mathfrak{M}_p$ which together span $\mathfrak{H}$. In that proof it is only necessary to separate the $\mathbf{R}_i$'s into p sets rather than 2. Then if E_i is the projection on $\mathfrak{M}_i$, $\sum_{n=1}^{p} E_n = 1$ and $D_{\mathbf{M}}(E_n) = \infty = D_{\mathbf{M}}(E_1)$.

Since $D_{\mathbf{M}}(E_t) = D_{\mathbf{M}}(E_1)$ there is a partially isometric $W_{t,1} \in \mathbf{M}$ with initial set $\mathfrak{M}_1$ and final set $\mathfrak{M}_t$ by the definition of the dimensionality function. If we now define $W_{t,s}$ by the equation $W_{t,s} = W_{t,1}W_{s,1}^*$ the properties of the partially isometric $W_{t,1}$ (cf. [5], §4.3) readily yield the equations of Def. 2.6.1 above. Hence the $W_{t,s}$ constitute a set of p matrix units with $E_0 = 1$, $W_{t,s} \in \mathbf{M}$.

LEMMA 2.6.3. *If* $\mathbf{M}$ *is in case* (I_n) *then* $\mathbf{M}$ *possesses a set of* p*-matrix units with* $E_0 = 1$ *if and only if* p *divides* n.

PROOF. Necessity: If $\mathbf{M}$ has a set of p-matrix units, with $E_0 = 1$, then by Lemma 2.6.1, the $W_{1,1}, \cdots, W_{p,p}$ are p pairwise orthogonal projections each with same relative dimension, and whose sum is 1. It follows that $D_{\mathbf{M}}(W_{t,t}) = (1/p)D_{\mathbf{M}}(\mathfrak{H})$. If we take $D_{\mathbf{M}}(\mathfrak{H}) = n$, we know that $D_{\mathbf{M}}(W_{t,t}) = (1/p)D_{\mathbf{M}}(\mathfrak{H})$ is a whole number and hence p divide n. (Cf. [5], Theorem VIII, p. 172.)

Sufficiency: If p divides n, we can find p dimensionally equivalent pairwise orthogonal projections $E_1, \cdots, E_p \in \mathbf{M}$ and with $\sum_{n=1}^{p} E_n = 1$. For if $D_{\mathbf{M}}(\mathfrak{H}) = n$, we can find a projection $E_1 \in \mathbf{M}$ with $D_{\mathbf{M}}(E_1) = n/p$ and we proceed as in the proof of Lemma 2.6.2 to find $E_2, \cdots, E_p$. The rest of that proof then applies to the present situation.

Theorem V yields

LEMMA 2.6.4. (i) *If* $\mathbf{M}$ *is a factor, not in case* (I_n) *and* p *is a finite integer, then there is an* $\mathbf{N}$ *such that* $\bar{\mathbf{N}}^p = \bar{\mathbf{M}}$.

(ii) *If* $\mathbf{M}$ *is in case* (I_n) *there is an* $\mathbf{N}$ *such that* $\bar{\mathbf{N}}^p = \bar{\mathbf{M}}$ *if and only if* p *divides* n.

(iii) *If there is an* $\mathbf{N}$ *such that* $\bar{\mathbf{N}}^p = \bar{\mathbf{M}}$ *for* $p < \infty$, $p = 2, 3, \cdots$, *then there is a manifold* $\mathfrak{M}$ *such that* $\overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{N}}$ *and* $D_{\mathbf{M}}(\mathfrak{M}) = (1/p)D_{\mathbf{M}}(\mathfrak{H})$.

As we remarked in footnote 15, we may add

LEMMA 2.6.5. *If* p *is finite,* $\mathbf{M}$ *is a factor and if* $\mathbf{N}$ *is such that* $\bar{\mathbf{N}}^p = \bar{\mathbf{M}}$ *then* $\bar{\mathbf{N}}$ *is uniquely determined.*

§2.7 It remains to consider the nature of $\mathbf{N}$ and also what happens if $p = \infty$.

LEMMA 2.7.1. *We have for any fixed* $p = 1, 2, \cdots, \infty$,

(i) $\bar{\mathbf{N}}$ *is a factor if and only if* $\bar{\mathbf{N}}^p$ *is one.*

(ii) *If* $\bar{\mathbf{N}}, \bar{\mathbf{N}}^p$ *are factors, then they belong to the same one of the Cases* (I), (II), (III).

(iii) *If* $\bar{\mathbf{N}}, \bar{\mathbf{N}}^p$, *are factors, then* $\bar{\mathbf{N}}^p$ *is in a finite case if and only if* $\bar{\mathbf{N}}$ *is in a finite case and* p *is finite.*

PROOF. Choose an $\mathbf{M}$ with $\bar{\mathbf{M}} = \bar{\mathbf{N}}^p$.

Ad (i). An $A^0 \in \mathbf{M}$ belongs to the center of $\mathbf{M}$ if and only if its matrix $\langle A_{t,s} \rangle$ (all $A_{t,s} \in \mathbf{N}$) commutes with all matrices $\langle B_{t,s} \rangle \in \mathfrak{J}$. For this it is necessary that it commute with all finite matrices $\langle B_{t,s} \rangle$ (all $B_{t,s} \in \mathbf{N}$, cf. B_p)). This is immediately seen to imply that $A_{t,s} = \delta_{t,s}A$ $(t, s \leqq p)$ for an A belonging to the center of $\mathbf{N}$. This condition is also clearly sufficient. Hence the A^0 of the center of $\mathbf{M}$ are precisely the $\alpha 1$ corresponding to the matrices $\langle \alpha\delta_{t,s}1 \rangle$ if and only if the A of the center of $\mathbf{N}$ are precisely the $\alpha 1$, i.e. $\mathbf{M}$ is a factor if and only if $\mathbf{N}$ is one.

This completes the proof.

Ad (ii). By Theorem V, $\mathbf{N}$ is isomorphic to an $\mathbf{M}_{(\mathfrak{M})}$. Hence our assertion follows from [5], p. 189, Lemma 11.4.3.

Ad (iii). The $W_{\mu,\mu}$ $(\mu \leqq p)$ are pairwise orthogonal projections $\in \mathbf{M}$ all with the same relative dimension in $\mathbf{M}$ owing to (i), (ii) in Lemma 2.6.1. Hence

$$D_{\mathbf{M}}(1) = \sum_{\mu=1}^{p} D_{\mathbf{M}}(W_{\mu,\mu}) = pD_{\mathbf{M}}(W_{1,1}). \tag{2.7.α}$$[16]

[16] Here as in some subsequent instances (proof of Lemma 5.3.4; III in §5.5) it is convenient to use the convention $\infty \cdot 0 = 0$.

Now $W_{1,1}$ is not zero, since if it were we should have $W_{\mu,\mu} = 0$ and $\sum_{\mu=1}^{p} W_{\mu,\mu} = 0$ for $p = 1, 2, \cdots, \infty$. Hence $D_{\mathfrak{M}}(W_{1,1}) \neq 0$. Now $\mathbf{M}$ is in a finite case if and only if $D_{\mathbf{M}}(1)$ is finite. By the above, this is equivalent to the statement that both p and $D_{\mathbf{M}}(W_{1,1})$ are finite. And the finiteness of $D_{\mathbf{M}}(W_{1,1}) = D_{\mathbf{M}}(\mathfrak{M})$, $(P_{\mathfrak{M}} = W_{1,1})$ is, according to [5], p. 189, Lemma 11.4.3, equivalent to $\mathbf{M}_{(\mathfrak{M})}$ being in a finite case, i.e. by Theorem V, to $\mathbf{N}$ being in a finite case.

This completes the proof.

§2.8 Let $\mathbf{M}$ be a factor in a Hilbert space $\mathfrak{H}$. As usual we denote by $D_{\mathbf{M}}(E)$ and by $D_{\mathbf{M}}(\mathfrak{M})$, $(E = P_{\mathfrak{M}} \epsilon \mathbf{M})$ a fixed relative dimension function of $\mathbf{M}$. We consider again the rings $\mathbf{M}^E$ and $\mathbf{M}_{(\mathfrak{M})}$ as described in the beginning of §2.5.

LEMMA 2.8.1. *If $\mathfrak{M}_1, \mathfrak{M}_2$ are two closed linear sets η $\mathbf{M}$ with $D_{\mathbf{M}}(\mathfrak{M}_1) = D_{\mathbf{M}}(\mathfrak{M}_2)$ then $\mathbf{M}_{(\mathfrak{M}_1)}$ and $\mathbf{M}_{(\mathfrak{M}_2)}$ (in $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively) are spatially isomorphic.*

PROOF. Since $D_{\mathbf{M}}(\mathfrak{M}_1) = D_{\mathbf{M}}(\mathfrak{M}_2)$ there exists a partially isometric $U \epsilon \mathbf{M}$ with the initial set $\mathfrak{M}_1$ and final set $\mathfrak{M}_2$. This is a one-to-one isomorphic mapping of $\mathfrak{M}_1$ on $\mathfrak{M}_2$ and it obviously carries $\mathbf{M}_{(\mathfrak{M}_1)}$ into $\mathbf{M}_{(\mathfrak{M}_2)}$. Hence it establishes the desired spatial isomorphism.

For the remainder of this chapter, we assume that the factor $\mathbf{M}$ is in a finite case.

Consider a real number α and a projection $E \epsilon \mathbf{M}$ or equivalently a closed linear set $\mathfrak{M}$ η $\mathbf{M}$ with $E = P_{\mathfrak{M}}$. We assume $E \neq 0$, i.e. $\mathfrak{M} \neq \{0\}$ and

$$\alpha = \frac{D_{\mathbf{M}}(E)}{D_{\mathbf{M}}(1)}. \tag{2.8.α}$$

Then the above Lemma 2.8.1 shows that $\overline{\mathbf{M}_{(\mathfrak{M})}}$ depends only on $\mathbf{M}$ and α but not on the E, $\mathfrak{M}$ themselves. (A spatial isomorphism implies an algebraic isomorphism.)

Consider now two algebraically isomorphic $\mathbf{M}_1$ and $\mathbf{M}_2$. Choose $E_1 \epsilon \mathbf{M}$ with $\alpha = D_{\mathbf{M}_1}(E)/D_{\mathbf{M}_1}(1)$. Then the isomorphism carries E_1 into an $E_2 \epsilon \mathbf{M}$ with $\alpha = D_{\mathbf{M}_2}(E_2)/D_{\mathbf{M}_2}(1)$. Let $\mathfrak{M}_1, \mathfrak{M}_2$ be η $\mathbf{M}_1$, η $\mathbf{M}_2$ respectively, and such that $P_{\mathfrak{M}_1} = E_1$, $P_{\mathfrak{M}_2} = E_2$. The isomorphism carries $\mathbf{M}_1^{E_1}$ into $\mathbf{M}_2^{E_2}$ in particular E_1 into E_2 and it leaves the operations αA, A^*, $A + B$, AB invariant. Hence it generates, by Lemma 2.5.1, an algebraic isomorphism of $(\mathbf{M}_1)_{(\mathfrak{M}_1)}$ and $(\mathbf{M}_2)_{(\mathfrak{M}_2)}$. Thus $\overline{\mathbf{M}_{(\mathfrak{M})}}$ (assuming its existence) is uniquely determined by $\overline{\mathbf{M}}$ and α. We denote this $\overline{\mathbf{M}_{(\mathfrak{M})}}$ by $\overline{\mathbf{M}}^{\alpha}$.

This notation could conflict with that of Theorem IV when $p = \alpha = 1$ But in that case clearly $\overline{\mathbf{M}}^1$ exists and is equal to $\overline{\mathbf{M}}$ under both definitions.

The α, for which $\overline{\mathbf{M}}^{\alpha}$ can be formed, comprise precisely the range of $D_{\mathbf{M}}(E)/D_{\mathbf{M}}(1)$ for all $E \epsilon \mathbf{M}$, $E \neq 0$. This is the set $(1/n, 2/n, \cdots, 1)$ if $\overline{\mathbf{M}}$ is in a case I_n and the set $0 < \alpha \leqq 1$ if $\overline{\mathbf{M}}$ is in a case (II_1).

Summing up,

THEOREM VI. *$\overline{\mathbf{M}}^{\alpha}$ exists if and only if α belongs to the following set: $(1/n, 2/n, \cdots, 1)$ if $\mathbf{M}$ is in a case (I_n), $(n = 1, 2, \cdots)$, $(0 < \alpha \leqq 1)$ if $\mathbf{M}$ is in the case (II_1).*

There is no conflict between this notation and that of Theorem IV.

Since the relative dimension $D_{\mathbf{M}}(E)$ can be characterized in a way which is

invariant under conjugate isomorphisms (cf. the remark at the end of §2.3), our present definition and that of §2.3 yield together

$$(\overline{\mathbf{M}}^c)^\alpha = (\overline{\mathbf{M}}^\alpha)^c \qquad (2.8.\beta)$$

(α in the range of Theorem VI).

Consider next two projections $E, F \in \mathbf{M}$, $E, F \neq 0$ and $E \leqq F$ or equivalently the closed linear sets $\mathfrak{M}, \mathfrak{N}$ η $\mathbf{M}$ with $P_{\mathfrak{M}} = E, P_{\mathfrak{N}} = F$. Hence $\mathfrak{M} \neq (0), \mathfrak{N} \neq (0)$ and $\mathfrak{M} \subseteq \mathfrak{N}$. To avoid misunderstanding, let 1 be the unit operator in $\mathfrak{H}$ and $1' = F_{(\mathfrak{N})}$ be the unit in $\mathfrak{N}$. Put

$$\alpha = \frac{D_{\mathbf{M}}(F)}{D_{\mathbf{M}}(1)}, \qquad \beta = \frac{D_{\mathbf{M}_{(\mathfrak{N})}}(E_{(\mathfrak{N})})}{D_{\mathbf{M}_{(\mathfrak{N})}}(1')} = \frac{D_{\mathbf{M}}(E)}{D_{\mathbf{M}}(F)}.$$

Clearly $\alpha\beta = D_{\mathbf{M}}(E)/D_{\mathbf{M}}(1)$ and $\mathbf{M}_{(\mathfrak{M})} = (\mathbf{M}_{(\mathfrak{N})})_{(\mathfrak{M})}$. Our definitions now yield immediately

$$(\overline{\mathbf{M}}^\alpha)^\beta = \overline{\mathbf{M}}^{\alpha\beta} \qquad (2.8.\gamma)$$

(α, β in the range of Theorem VI).

§2.9 We continue the discussion of the preceding section

LEMMA 2.9.1. *If p is finite and $\mathbf{M}$ and $\mathbf{N}$ are factors in a finite case, $\overline{\mathbf{N}} = \overline{\mathbf{M}}^{1/p}$ is equivalent to the statement $\overline{\mathbf{M}} = \overline{\mathbf{N}}^p$. (The statement "$\overline{\mathbf{M}}^{1/p} = \overline{\mathbf{N}}$" includes "$\overline{\mathbf{M}}^{1/p}$ exists.")*

PROOF. Assume $\overline{\mathbf{M}} = \overline{\mathbf{N}}^p$. It follows from Theorem V that there is an E with $D_{\mathbf{M}}(E) = (1/p)D_{\mathbf{M}}(1)$ such that $\overline{\mathbf{N}} = \overline{\mathbf{M}}_{(\mathfrak{M})}$ where $\mathfrak{M}$ is such that $E = P_{\mathfrak{M}}$. Hence, by the above definition, $\overline{\mathbf{M}}^{1/p} = \overline{\mathbf{N}}$.

Conversely, suppose $\overline{\mathbf{M}}^{1/p} = \overline{\mathbf{N}}$. It follows that $1/p$ is in the set of Theorem VI and hence, if $\mathbf{M}$ is in a case (I_n) p divides n. Thus Lemmas 2.6.2 and 2.6.3 imply that there is an $\overline{\mathbf{N}}_0$ such that $\overline{\mathbf{N}}_0^p = \overline{\mathbf{M}}$. Hence the above result (with $\overline{\mathbf{N}}_0$ in place of $\overline{\mathbf{N}}$) gives $\overline{\mathbf{N}}_0 = \overline{\mathbf{M}}^{1/p}$ i.e. $\overline{\mathbf{N}}_0 = \overline{\mathbf{N}}$. Consequently $\overline{\mathbf{N}}^p = \overline{\mathbf{N}}_0^p = \overline{\mathbf{M}}$.

This completes the proof.

LEMMA 2.9.2. *If $\overline{\mathbf{M}}^\alpha$ exists (in the sense of Theorem* VI*) and if $p\alpha = q\beta$, $(p, q = 1, 2, \cdots)$ then $(\overline{\mathbf{M}}^\alpha)^p = (\overline{\mathbf{M}}^q)^\beta$. (This is to include the statement that $(\overline{\mathbf{M}}^q)^\beta$ exists.)*

PROOF. $p\alpha = q\beta$. Hence $\alpha/q = \beta/p$, $\overline{\mathbf{M}}^\alpha$ exists. From this we shall draw inferences so that the existence of each expression that appears will be established when we write it down. In this way (2.8.γ) and Lemma 2.9.1 give

$$\overline{\mathbf{M}}^\alpha = ((\overline{\mathbf{M}}^q)^{\frac{1}{q}})^\alpha = (\overline{\mathbf{M}}^q)^{\frac{\alpha}{q}} = (\overline{\mathbf{M}}^q)^{\frac{\beta}{p}}$$

Hence by Lemma 2.9.1,

$$(\overline{\mathbf{M}}^\alpha)^p = ((\overline{\mathbf{M}}^q)^{\frac{\beta}{p}})^p = (((\overline{\mathbf{M}}^q)^\beta)^{\frac{1}{p}})^p = (\overline{\mathbf{M}}^q)^\beta$$

LEMMA 2.9.3. *For a given $\overline{\mathbf{M}}$ consider all α of the set in Theorem* VI*, and all $p = 1, 2, \cdots$. Then we have*

(i) *The range of $\theta = p\alpha$ is the set $(1/n, 2/n, \cdots)$ if $\mathbf{M}$ is in a case (I_n) $(n = 1, 2, \cdots)$ and the set $0 < \theta < \infty$ if $\mathbf{M}$ is in the case (II_1).*

(ii) $(\overline{\mathbf{M}}^{\alpha})^{p}$ *depends only on* $\theta = p\alpha$ *and not on the individual values of* α *and* p.

PROOF. Ad (i). Immediate by Theorem V.

Ad (ii). Lemma 2.9.2 yields, when $p\alpha = q\beta$, $(\overline{\mathbf{M}}^{\alpha})^{p} = (\overline{\mathbf{M}}^{q})^{\beta}$ and $(\overline{\mathbf{M}}^{\beta})^{q} = (\overline{\mathbf{M}}^{q})^{\beta}$. Hence $(\overline{\mathbf{M}}^{\alpha})^{p} = (\overline{\mathbf{M}}^{\beta})^{q}$.

We denote the above $(\overline{\mathbf{M}}^{\alpha})^{p}$, $\theta = p\alpha$ by $\overline{\mathbf{M}}^{\theta}$.

This notation does not conflict with those of Theorems IV and VI. This can be seen by putting $\alpha = 1$ or $p = 1$, respectively.

Summing up,—

THEOREM VII. $\overline{\mathbf{M}}^{\theta}$ *exists if and only if* θ *belongs to the following set*: $(1/n, 2/n, \cdots)$ *if* $\mathbf{M}$ *is in a case* (I_n), $(n = 1, 2, \cdots)$, $(0 < \theta < \infty)$ *if* $\mathbf{M}$ *is in the case* (II_1).

There is no conflict between this notation and that of Theorems IV and VI.

Combining (2.4.α) and (2.8.β) gives

$$(\overline{\mathbf{M}}^{c})^{\theta} = (\overline{\mathbf{M}}^{\theta})^{c} \qquad (\theta \text{ in the set of Theorem VII}). \tag{2.9.α}$$

Also, by the use of (2.4.β), (2.8.γ) and Lemma 2.9.2, we obtain

$$(((\overline{\mathbf{M}}^{\alpha})^{p})^{\beta})^{q} = (((\overline{\mathbf{M}}^{\alpha})^{\beta})^{p})^{q} = (\overline{\mathbf{M}}^{\alpha\beta})\mathbf{M}^{pq}.$$

Hence if we put $\theta = p\alpha$, $\xi = q\beta$, then

$$(\overline{\mathbf{M}}^{\theta})^{\xi} = (\overline{\mathbf{M}}^{\xi})^{\theta} = \overline{\mathbf{M}}^{\xi\theta} \qquad (\theta \text{ and } \xi \text{ in the set of Theorem VII}). \tag{2.9.β}$$

(Concerning the exponent ∞, consider the remark at the end of §2.5.)

§2.10. Denote the set of all θ for which $\overline{\mathbf{M}}^{\theta}$ exists and equals $\overline{\mathbf{M}}$ by

$$\mathfrak{G} = \mathfrak{G}(\overline{\mathbf{M}}). \tag{2.10.α}$$

Obviously $1 \in \mathfrak{G}$ and $\theta, \xi \in \mathfrak{G}$ imply $\theta\xi \in \mathfrak{G}$. Since $(\overline{\mathbf{M}}^{\theta})^{1/\theta}$ exists and equals $\overline{\mathbf{M}}$, $\overline{\mathbf{M}}^{\theta} = \overline{\mathbf{M}}$ implies $\overline{\mathbf{M}}^{1/\theta} = \overline{\mathbf{M}}$ i.e. $\theta \in \mathfrak{G}$ implies $1/\theta \in \mathfrak{G}$. (All this is due to (2.9.β) in §2.9.)

THEOREM VIII. *The set* $\mathfrak{G}$ *of* (2.10.α) *above is a subgroup of* P *the multiplication group of the real numbers* θ, $0 < \theta < \infty$. *We call* $\mathfrak{G}$ *the fundamental group of* $\overline{\mathbf{M}}$.

Some properties of $\mathfrak{G}$.

LEMMA 2.10.1. $\overline{\mathbf{M}}^{\theta} = \overline{\mathbf{M}}^{\xi}$ (*both sides are assumed to exist*) *is equivalent to* $\theta/\xi \in \mathfrak{G}$.

PROOF. Sufficiency: $\theta/\xi \in \mathfrak{G}$ implies $\overline{\mathbf{M}}^{\theta/\xi} = \overline{\mathbf{M}}$. Hence $\overline{\mathbf{M}}^{\theta} = (\overline{\mathbf{M}}^{\theta/\xi})^{\xi} = \overline{\mathbf{M}}^{\xi}$.

Necessity: Assume $\overline{\mathbf{M}}^{\theta} = \overline{\mathbf{M}}^{\xi}$. $(\overline{\mathbf{M}}^{\xi})^{1/\xi}$ exists and equals $\overline{\mathbf{M}}$. Hence $(\overline{\mathbf{M}}^{\theta})^{1/\xi} = \overline{\mathbf{M}}^{\theta/\xi}$ exists and equals $\overline{\mathbf{M}}$. Thus $\theta/\xi \in \mathfrak{G}$.

LEMMA 2.10.2. $\overline{\mathbf{M}}$, $\overline{\mathbf{M}}^{c}$ *and all* $\overline{\mathbf{M}}^{\theta}$ *have the same fundamental group.*

PROOF. Ad $\overline{\mathbf{M}}^{c}$. Immediate by (2.9.$\alpha$).

Ad. $\overline{\mathbf{M}}^{\theta}$. Immediate by Lemma 2.10.1.

Before we conclude this chapter, we determine the behavior of the discrete finite cases, i.e. the (I_n), $n = 1, 2, \cdots$.

Consider the ring $\mathbf{\Theta}$ of all operators $\alpha \cdot 1$. This is obviously the prototype of all factors of the case (I_1), its type $\overline{\mathbf{\Theta}}$ is that of the ring of all complex numbers.

Now we have

LEMMA 2.10.3. *If* $\mathbf{M}$ *is in case* (I_n), $(n = 1, 2, \cdots)$ *then* $\overline{\mathbf{M}}^\alpha$ *(*α *finite), exists if and only if* $n\alpha = 1, 2, \cdots$, *and then* $\overline{\mathbf{M}}^\alpha = \overline{\Theta}^{n\alpha}$.

PROOF. The range asserted for α is the same as that of Theorem VII. Put $n\alpha = k$. Then

$$\overline{\mathbf{M}}^\alpha = \overline{\mathbf{M}}^{\frac{k}{n}} = \left(\overline{\mathbf{M}}^{\frac{1}{n}}\right)^k.$$

Now consider $\overline{\mathbf{M}}^{1/n}$. Apply the definition of §2.8, preceding Theorem VII. For the E in question, $D_{\mathbf{M}}(E)/D_{\mathbf{M}}(1)$ assumes its minimal value, $1/n$. Hence, this E is minimal. (Cf. [5], pp. 143–144, Def. 5.1.2.) Consequently $\mathbf{M}^E$ is the set of all αE (cf. [5], p. 144, the last part of the proof of Lemma 5.1.3—or it may be proved directly). Hence $\mathbf{M}_{(\mathfrak{M})}$ is the set of all $\alpha \cdot 1$ in $\mathfrak{M}$ i.e.

$$\overline{\mathbf{M}}^{\frac{1}{n}} = \overline{\mathbf{M}}_{(\mathfrak{M})} = \overline{\Theta}.$$

Consequently

$$\overline{\mathbf{M}}^\alpha = \overline{\Theta}^k, \qquad k = n\alpha$$

as desired.

COROLLARY. *If* $\mathbf{M}$ *is in a case* $(I_n)(n = 1, 2, \cdots)$ *then* $\overline{\mathbf{M}} = \Theta^n$.

PROOF. Put $\alpha = 1$ in the above lemma.

LEMMA 2.10.4. *If* $\mathbf{M}$ *is in a case* $(I_n)(n = 1, 2, \cdots)$ *then* $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$ *and* $\mathfrak{G} = \mathfrak{G}(\mathbf{M}) = (1)$.

PROOF.[17] Ad. $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$. Clearly $\overline{\Theta}^c = \Theta$. Hence the corollary to Lemma 2.10.3, and (2.4.α) or (2.9.α) imply $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$.

Ad. $\mathfrak{G} = (1)$. By Lemma 2.10.3 and its corollary, $\overline{\mathbf{M}}^\alpha$ is in case (I_k), $k = n\alpha$. Hence $\overline{\mathbf{M}}^\alpha = \overline{\mathbf{M}}$ implies $n\alpha = n$ or $\alpha = 1$. So $\mathfrak{G} = (1)$.

The validity or invalidity of the equation $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$ and the fundamental group $\mathfrak{G} = \mathfrak{G}(\overline{\mathbf{M}})$ are algebraical isomorphism invariants of $\mathbf{M}$. Lemma 2.10.4 showed us how they behave when $\mathbf{M}$ is in a discrete finite case, i.e. in a case (I_n), $n = 1, 2, \cdots$. The really interesting factors are the remaining finite ones, those in the continuous finite case (II_1). We shall see that they are not all isomorphic to each other, but we shall achieve this differentiation with the help of other invariants. (Cf. the end of §5.6 and the beginning of §6.1.)

For all $\overline{\mathbf{M}}$ in case (II_1) for which we have succeeded in settling this question, $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$ and $\overline{\mathbf{M}}^\theta = \overline{\mathbf{M}}$ (θ in the set of Theorem VII), i.e. $\mathfrak{G} = P$, was found. (Cf. Theorem XV and the end of §5.6. Observe Lemma 2.10.4 by comparison.) There seems to be, however, no reason to believe the general validity of these relations. The general behavior of the above invariants remains therefore an open question.

CHAPTER III. CHARACTERIZATION OF ALL SPATIAL TYPES IN TERMS OF ALGEBRAICAL TYPES

§3.1 We have classified all operator rings $\mathbf{M}$ according to their types $\overline{\mathbf{M}}$ i.e. with respect to algebraical isomorphism. Now we shall introduce a broader

[17] Both assertions could be proved directly by matrix considerations.

classification, to be called the *genus*, each genus consisting of one or more types. It will be necessary, however, to restrict this new classification to factors $\mathbf{M}$.

We need some auxiliary lemmas which lead up to the desired definition. Throughout this section $\mathbf{M}$ will be a factor in a Hilbert space $\mathfrak{H}$. As before, we denote by $D_{\mathbf{M}}(E)$ and $D_{\mathbf{M}}(\mathfrak{M})$ a fixed relative dimension function of $\mathbf{M}$ applicable to the $E \in \mathbf{M}$ or the $\mathfrak{M} \eta \mathbf{M}$.

LEMMA 3.1.1. *If $\mathfrak{M}$ is a closed linear set, $\mathfrak{M} \eta \mathbf{M}$ and such that $\mathfrak{M} \neq (0)$ and $D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{M}) = \infty$ then*

$$\bar{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty}.$$

PROOF. Under our hypothesis, $\mathfrak{M}$ "divides" $\mathfrak{H} \ominus \mathfrak{M}$ an infinite number of times, i.e. there are an infinite number of pairwise orthogonal closed linear sets $\mathfrak{M}_2', \mathfrak{M}_3', \cdots$ such that $D_{\mathbf{M}}(\mathfrak{M}_i') = D_{\mathbf{M}}(\mathfrak{M})$ and $\mathfrak{M}_i' \subset \mathfrak{H} \ominus \mathfrak{M}$ for $i = 2, 3, \cdots$. If $\mathfrak{M}$ is relatively finite-dimensional, this is obvious. If $\mathfrak{M}$ is infinite-dimensional, we apply a variant of [5], Lemma 7.2.3, p. 157, to $\mathfrak{H} \ominus \mathfrak{M}$ which shows that $\mathfrak{H} \ominus \mathfrak{M}$ is determined by a denumerably infinite number of pairwise orthogonal closed linear sets $\mathfrak{M}_2', \mathfrak{M}_3', \cdots$, each of which is relatively infinite-dimensional. (To prove the variant, it is only necessary, in the proof of [5], Lemma 7.2.3, to divide the $\mathfrak{P}_1, \mathfrak{P}_2, \cdots$, into an infinite number of sets, each having an infinite number of $\mathfrak{P}_i$'s, rather than into two sets.)

If we now apply [5], Lemma 7.1.2, p. 155, to $\mathfrak{M}$ and $\mathfrak{H} \ominus \mathfrak{M}$ we see that $\mathfrak{H} \ominus \mathfrak{M}$ is determined by a denurably infinite number of sets $\mathfrak{M}_2, \mathfrak{M}_3, \cdots$, which are pairwise orthogonal and for which $D_{\mathbf{M}}(\mathfrak{M}_i) = D_{\mathbf{M}}(\mathfrak{M})$ for $i = 2, 3, \cdots$. Let $\mathfrak{M} = \mathfrak{M}_1$.

The second paragraph of the proof of Lemma 2.6.2 can now be used to show the existence in $\mathbf{M}$ of a set of ∞-matrix units $W_{\mu,\nu}$ with $W_{\mu,\mu} = P_{\mathfrak{M}_\mu}$, $W_{1,1} = P_{\mathfrak{M}_1} = P_{\mathfrak{M}}$. For the hypothesis $p < \infty$ is not used here. Furthermore, $E_0 = \sum_{\mu=1}^{\infty} W_{\mu,\mu} = W_{1,1} + \sum_{\mu=2}^{\infty} W_{\mu,\mu} = P_{\mathfrak{M}_1} + \sum_{\mu=2}^{\infty} P_{\mathfrak{M}_\mu} = P_{\mathfrak{M}} + P_{\mathfrak{H} \ominus \mathfrak{M}} = 1$. Thus Theorem V in §2.6 above yields the desired result.

LEMMA 3.1.2. *If $\mathfrak{M}$ is a closed linear set, $\mathfrak{M} \eta \mathbf{M}$ and $\neq (0)$ and if $\mathbf{M}$ is in an infinite case, then*

$$\bar{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty}.$$

PROOF. If $D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{M}) = \infty$ the result is implied by Lemma 3.1.1. Hence we may assume that $D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{M})$ is finite. Since $\mathbf{M}$ is in an infinite case, $D_{\mathbf{M}}(\mathfrak{H})$ is infinite and hence $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{M})$ is infinite also.

Since $D_{\mathbf{M}}(\mathfrak{H})$ is infinite, there exists by [5], Lemma 2.7.3, p. 157, a closed linear set $\mathfrak{N}$ such that $D_{\mathbf{M}}(\mathfrak{N})$ and $D_{\mathbf{M}}(\mathfrak{H} \ominus \mathfrak{N})$ are also infinite. Hence $\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}_{(\mathfrak{N})}}$ by lemma 2.8.1 (we need only the algebraical, not the spatial, isomorphism) and $\bar{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{N})}})^{\infty}$ by Lemma 3.1.1. These give together $\bar{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty}$.

LEMMA 3.1.3. *A factor $\mathbf{M}$ is in an infinite case if and only if*

$$\bar{\mathbf{M}} = \bar{\mathbf{M}}^{\infty}.$$

PROOF. Sufficiency: Immediate by (iii) in Lemma 2.7.1.

Necessity: Put $\mathfrak{M} = \mathfrak{H}$ in Lemma 3.1.2.

DEFINITION 3.1.1. *If, for two factors,* $\mathbf{M}$, $\mathbf{N}$

$$\bar{\mathbf{N}} = \overline{\mathbf{M}_{(\mathfrak{M})}}$$

for a suitable closed set $\mathfrak{M} \neq (0)$ *and* $\mathfrak{M}\ \eta\ \mathbf{M}$ *in the Hilbert space* $\mathfrak{H}$ *of* $\mathbf{M}$ *then we say that* $\bar{\mathbf{M}}$ *is a multiple of* $\bar{\mathbf{N}}$, *or that* $\bar{\mathbf{N}}$ *is a divisor of* $\bar{\mathbf{M}}$.

It is clear that this relationship concerns $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$. Cf. the argument immediately preceding Theorem VI in §2.8. It is also obviously transitive.

LEMMA 3.1.4. $\bar{\mathbf{N}}$ *is a divisor of* $\bar{\mathbf{M}}$ *if and only if* $\alpha)\bar{\mathbf{M}} = \bar{\mathbf{N}}$ *when* $\bar{\mathbf{N}}$ *is in an infinite case* $\beta)\ \bar{\mathbf{M}} = \bar{\mathbf{N}}^{\theta}$ *with a* $\theta \geqq 1$ *in the range of Theorem* VII *or of Theorem* IV′ (i.e., $\theta = \infty$) *when* $\bar{\mathbf{N}}$ *is in a finite case.*

PROOF. Ad α). Sufficiency: Obvious.

Necessity: Since $\bar{\mathbf{N}} = \overline{\mathbf{M}_{(\mathfrak{M})}}$ is in an infinite case, $D_{\mathbf{M}}(\mathfrak{M})$ is infinite by [5], p. 189, Lemma 11.4.3. Hence $\mathbf{M}$ is in an infinite case and thus Lemma 3.1.2 gives $\bar{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty} = \bar{\mathbf{N}}^{\infty}$. On the other hand, $\bar{\mathbf{N}} = \bar{\mathbf{N}}^{\infty}$ by Lemma 3.1.3. Thus $\bar{\mathbf{M}} = \bar{\mathbf{N}}$.

Ad β). Sufficiency: θ finite: Then $\bar{\mathbf{N}} = \bar{\mathbf{M}}^{\alpha}$ with $\alpha = 1/\theta < 1$. Hence the assertion follows by our definition of $\bar{\mathbf{M}}^{\alpha}$ for $\alpha \leqq 1$ in §2.8. θ infinite: Immediate by Theorem V.

Necessity: $\bar{\mathbf{M}}$ in a finite case: Clearly $\bar{\mathbf{N}} = \overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{M}}^{\alpha}$ with $\alpha = D_{\mathbf{M}}(E)/D_{\mathbf{M}}(1) \leqq 1$, $(E = P_{\mathfrak{M}})$. Hence $\bar{\mathbf{M}} = \bar{\mathbf{N}}^{\theta}$ with $\theta = 1/\alpha \geqq 1$. $\bar{\mathbf{M}}$ in an infinite case: $\bar{\mathbf{M}} = (\overline{\mathbf{M}_{(\mathfrak{M})}})^{\infty} = \bar{\mathbf{N}}^{\infty}$ by Lemma 3.1.2.

LEMMA 3.1.5. *The four following statements are equivalent to each other:*

$\alpha)$ $\bar{\mathbf{M}}$ *is either a multiple or divisor of* $\bar{\mathbf{N}}$,

$\beta)$ $\bar{\mathbf{M}}$ *and* $\bar{\mathbf{N}}$ *possess a common multiple,*

$\gamma)$ $\bar{\mathbf{M}}$ *and* $\bar{\mathbf{N}}$ *possess a common divisor,*

$\delta)$ $\bar{\mathbf{M}}^{\infty} = \bar{\mathbf{N}}^{\infty}$.

PROOF. The implications

$$\alpha) \rightarrow \beta), \qquad \alpha) \rightarrow \gamma)$$

are obvious. Furthermore

$$\delta) \rightarrow \beta)$$

follows from Theorem V, since this theorem shows that $\bar{\mathbf{M}}^{\infty} = \bar{\mathbf{N}}^{\infty}$ is a multiple of $\bar{\mathbf{M}}$ and of $\bar{\mathbf{N}}$.

We prove next

$$\beta) \rightarrow \alpha)$$

Let $\bar{\mathbf{L}}$ be the common multiple of both $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$. Let closed linear sets $\mathfrak{M}$ and $\mathfrak{N}$ be chosen in the $\mathfrak{H}$ associated with $\mathbf{L}$, so that $\bar{\mathbf{M}} = \overline{\mathbf{L}_{(\mathfrak{M})}}$ and $\bar{\mathbf{N}} = \overline{\mathbf{L}_{(\mathfrak{N})}}$. Since $\mathbf{L}$ is a factor either $\mathfrak{M}$ has the same relative dimension (in $\mathbf{L}$) as an $\mathfrak{N}' \subseteq \mathfrak{N}$ or $\mathfrak{N}$ has the same relative dimension as an $\mathfrak{M}' \subset \mathfrak{M}$. Cf. [5], p. 153, Lemma 6.2.3, or p. 154, Theorem VI. By symmetry we may assume the former. By Lemma 2.8.1 we may now replace $\mathfrak{M}$ by $\mathfrak{N}'$, *i.e.* we can assume that $\mathfrak{M} \subseteq \mathfrak{N}$.

Thus $\overline{(\mathbf{L}_{(\mathfrak{R})(\mathfrak{M})})} = \overline{\mathbf{L}_{(\mathfrak{M})}}$ so that $\overline{\mathbf{L}_{(\mathfrak{R})}}$ is a multiple of $\overline{\mathbf{L}_{(\mathfrak{M})}}$. Hence $\bar{\mathbf{N}}$ is a multiple of $\bar{\mathbf{M}}$.

Finally

$$\gamma) \rightarrow \delta)$$

Let $\bar{\mathbf{K}}$ be a common divisor of $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$. Since $\bar{\mathbf{M}}$ is a divisor of $\bar{\mathbf{M}}^\infty$ by Theorem V, so $\bar{\mathbf{K}}$ is a divisor of $\bar{\mathbf{M}}^\infty$ too. Now $\bar{\mathbf{M}}^\infty$ is in an infinite case by (iii) in Lemma 2.7.1. Hence $\bar{\mathbf{K}}^\infty = \bar{\mathbf{M}}^\infty$ by Lemma 3.1.2. Similarly $\bar{\mathbf{K}}^\infty = \bar{\mathbf{N}}^\infty$. Consequently $\bar{\mathbf{M}}^\infty = \bar{\mathbf{N}}^\infty$.

All these equivalences give together

$$\alpha) \rightarrow \gamma) \rightarrow \delta) \rightarrow \beta) \rightarrow \alpha)$$

thus completing the proof.

DEFINITION 3.1.2. *If the equivalent conditions of Lemma* 3.1.5 *are satisfied then we say that* $\bar{\mathbf{M}}$, $\bar{\mathbf{N}}$ *are commensurable.*

$\delta)$ of Lemma 3.1.5 implies that the notion of commensurability is reflexive, symmetric and transitive.

LEMMA 3.1.6. $\bar{\mathbf{M}}$, $\bar{\mathbf{N}}$ *are commensurable if and only if*

$\alpha)$ $\bar{\mathbf{M}} = \bar{\mathbf{N}}$ *when* $\bar{\mathbf{M}}$, $\bar{\mathbf{N}}$ *are both in infinite cases.*

$\beta)$ $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\infty$ *when* $\bar{\mathbf{M}}$ *is in an infinite case and* $\bar{\mathbf{N}}$ *in a finite case*

$\gamma)$ $\bar{\mathbf{M}}^\infty = \bar{\mathbf{N}}$ *when* $\bar{\mathbf{M}}$ *is in a finite case and* $\bar{\mathbf{N}}$ *in an infinite case*

$\delta)$ $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\theta$ *with a* θ *in the range of Theorem* VII *when* $\bar{\mathbf{N}}$ *and* $\bar{\mathbf{M}}$ *are both in finite cases.*

PROOF. Ad $\alpha) - \gamma)$. Condition δ of Lemma 3.1.5 for commensurability is $\bar{\mathbf{M}}^\infty = \bar{\mathbf{N}}^\infty$. Lemma 3.1.3 shows that when $\bar{\mathbf{M}}$ is in an infinite case we may substitute $\bar{\mathbf{M}}$ for $\bar{\mathbf{M}}^\infty$ in this condition and correspondingly for $\bar{\mathbf{N}}$. When $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ are both infinite, this yields $\alpha)$ while if only one is infinite we obtain $\beta)$ or $\gamma)$

Ad $\delta)$. By Lemma 3.1.4, $\beta)$, $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\theta$ for $\theta \geqq 1$ in the range of Theorem VII is equivalent to $\bar{\mathbf{N}}$ a divisor of $\bar{\mathbf{M}}$. Since $\bar{\mathbf{M}}$ is finite, θ is finite by Lemma 2.7.1 (iii).

On the other hand, $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\theta$ for a $\theta \leqq 1$ in the range of Theorem VII (for $\bar{\mathbf{N}}$) is equivalent to $\bar{\mathbf{M}}^{1/\theta} = \bar{\mathbf{N}}$ with $1/\theta$ in the range of Theorem VII (for $\mathbf{M}$). (Cf. (2.9.β). That θ is in the correct sets in the discrete cases is readily shown by reference to Lemma 2.10.3 and its corollary.) Lemma 3.1.4 with $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ interchanged shows that this is equivalent to $\bar{\mathbf{N}}$ is a multiple of $\bar{\mathbf{M}}$.

Thus $\bar{\mathbf{M}} = \bar{\mathbf{N}}^\theta$ for a θ in the range of Theorem VII is equivalent to $\bar{\mathbf{N}}$ is a divisor of $\bar{\mathbf{M}}$ or $\bar{\mathbf{N}}$ is a multiple of $\mathbf{M}$, i.e. to the condition of Lemma 3.1.5 for commensurability.

We call the abstraction of type $\bar{\mathbf{M}}$ (for factors) with respect to commensurability its *genus*, i.e. two types $\bar{\mathbf{M}}_1$ and $\bar{\mathbf{M}}_2$ will be said to have the same genus if and only if they are commensurable. We denote the genus of the type $\bar{\mathbf{M}}$ by $\bar{\mathbf{M}}$. Notions invariant under commensurability will be said to be notions concerning the genus $\bar{\mathbf{M}}$ (and not $\bar{\mathbf{M}}$ or $\mathbf{M}$ itself).

Thus the cases I, II, III are notions concerning the genus. This follows from (ii) in Lemma 2.7.1 (with $p = \infty$) and the characterization $\delta)$ in Lemma 3.1.5.

Similarly $\bar{\mathbf{M}}^c$ is an operation concerning the genus. This follows from (2.5. ϵ and the above-mentioned characterization. Therefore we can define

$$\bar{\bar{\mathbf{M}}}^c = \overline{\bar{\mathbf{M}}^c}.$$

Observe finally that $(\overline{\mathbf{M}_{(\mathfrak{M})}})$, $\bar{\mathbf{M}}^p$, $p = 1, 2, \cdots, \infty$ and (when it is defined, i.e. for $\mathbf{M}$ in a finite case) $\bar{\mathbf{M}}^\theta$ (θ in the range of Theorem VII) have the same genus as $\mathbf{M}$ (for $(\overline{\mathbf{M}_{(\mathfrak{M})}})$ this follows from α) in Lemma 3.1.5; for $\bar{\mathbf{M}}^p$ this is implied by Theorem V and Lemma 3.1.5 α); and for $\bar{\mathbf{M}}^\theta$ this follows from Lemma 3.1.6δ).)

It is clear then that being in a finite or an infinite case does not concern the genus (cf. Theorem IX for details).

§3.2 Since the cases (I), (II), (III) are notions concerning the genus, each one of these cases is the sum of one or more genera. And of course each genus is the sum of one or more types. These are the details.

THEOREM IX. α) *Case* (I) *consists of exactly one genus. This genus contains exactly one type in an infinite case*: (I_∞); *and for every positive integer* M *exactly one type* (I_n), *these being all its types in finite cases.*

β) *Case* (II) *consists of one or more genera. Each such genus contains exactly one type in an infinite case* (II_∞) *and one or more types in finite cases* (II_1).

γ) *Case* (III) *consists of one or more genera. Each such genus contains exactly one type which is, of course, in an infinite case* (III_∞).

PROOF. Ad α). Each one of the cases (I_1), (I_2), $\cdots$, (I_∞) contains exactly one type by [5], p. 173, Lemma 8.6.1. They can obviously all be obtained from the last one by the operations $\mathbf{M}_{(\mathfrak{M})}$. Hence they are all of the same genus as this last by α) of Lemma 3.1.5 and Def. 3.1.1.

Ad β). Consider a genus $\bar{\mathbf{M}}$ of case (II). Then there exists an $\mathbf{M}_{(\mathfrak{M})}$ in a *finite* case, for any $\mathfrak{M}$ with $D_{\mathbf{M}}(\mathfrak{M}) < \infty$ will do. Thus $\overline{\mathbf{M}_{(\mathfrak{M})}}$ is finite and it belongs to the given genus of $\mathbf{M}$ by condition α) of Lemma 3.1.5. $\bar{\mathbf{M}}^\infty$ belongs also to this genus for the same reason, since $\bar{\mathbf{M}}$ is a divisor of $\bar{\mathbf{M}}^\infty$ by Theorem V of §2.6. The infinite case in a particular genus is unique by α) of Lemma 3.1.6.

Ad γ). Consider a genus $\bar{\mathbf{M}}$ of case (III_∞). Then $\bar{\mathbf{M}}$ must be in an infinite case, and consequently it is unique by α) of Lemma 3.1.6.

Observe that none of the three cases (I), (II) or (III) is empty. Cf. [8], p. 94, §1, in the introduction.

LEMMA 3.2.1. *Consider a genus* $\bar{\mathbf{M}}$ *which contains types* $\bar{\mathbf{M}}$ *in finite cases, i.e. one in the cases* (I), (II) (*cf. Theorem* IX *above*). *Then the fundamental group* $\mathfrak{G}(\bar{\mathbf{M}})$ *is a notion concerning the genus* $\bar{\mathbf{M}}$, *i.e. it is the same for all the types* $\bar{\mathbf{M}}$ *of the genus which are in a finite case.*

PROOF. Immediate by δ) in Lemma 3.1.6 and Lemma 2.10.2.

We shall therefore denote the fundamental group from now on by $\mathfrak{G}(\bar{\mathbf{M}})$. Some simple properties of $\mathfrak{G}(\bar{\mathbf{M}})$ follow.

LEMMA 3.2.2. *Consider a genus* $\bar{\mathbf{M}}$ *as above. Form the ring* $\mathbf{M}$ *and two closed linear sets,* $\mathfrak{M}$, $\mathfrak{N}$ η $\mathbf{M}$ *which are both* $\neq (0)$ *and of finite relative dimension. Then we have*

(i) $$\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}_{(\mathfrak{N})}^{D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}}(\mathfrak{N})}},$$

(ii) $$\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}_{(\mathfrak{N})}} \quad \textit{if and only if} \quad D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}}(\mathfrak{N}) \in \mathfrak{G}(\bar{\mathbf{M}}).$$

PROOF. Ad (i). Suppose first that $D_{\mathbf{M}}(\mathfrak{M}) \leqq D_{\mathbf{M}}(\mathfrak{N})$.

Here $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{M}')$ for an $\mathfrak{M}' \subset \mathfrak{N}$ and since by Lemma 2.8.1, $\overline{\mathbf{M}_{(\mathfrak{M}')}} = \overline{\mathbf{M}_{(\mathfrak{M})}}$ we may suppose $\mathfrak{M} \subset \mathfrak{N}$. If $\mathfrak{P}$ is a set $\eta\,\mathbf{M}$ and $\subset \mathfrak{N}$, $D_{\mathbf{M}}(\mathfrak{P})$ is a dimension function for $\mathbf{M}_{(\mathfrak{N})}$. Since $\mathfrak{M} \subsetneqq \mathfrak{N}$ by definition, $\overline{\mathbf{M}_{(\mathfrak{M})}} = (\overline{\mathbf{M}_{(\mathfrak{N})}})^{\alpha}$ for $\alpha = D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}}(\mathfrak{N})$. (Cf. §2.8, $\mathfrak{H}$, $\mathbf{M}$ are replaced by $\mathfrak{N}$, $\mathbf{M}_{(\mathfrak{N})}$.)

On the other hand, if $D_{\mathbf{M}}(\mathfrak{M}) > D_{\mathbf{M}}(\mathfrak{N})$ the above argument yields $(\overline{\mathbf{M}_{(\mathfrak{M})}})^{1/\alpha} = \overline{\mathbf{M}_{(\mathfrak{N})}}$. Equation (2.9.$\beta$) can now be used to show that $\overline{\mathbf{M}_{(\mathfrak{M})}} = (\overline{\mathbf{M}_{(\mathfrak{N})}})^{\alpha}$, suitable use of Lemma 2.10.3 and its corollary being made in the discrete case.

Ad (ii). By the definition of $\mathfrak{G}(\overline{\mathbf{M}_{(\mathfrak{N})}})$, (i) implies $\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}_{(\mathfrak{N})}}$ if and only if $\alpha \in \mathfrak{G}(\overline{\mathbf{M}_{(\mathfrak{N})}})$. But $\mathfrak{G}(\overline{\mathbf{M}_{(\mathfrak{N})}}) = \mathfrak{G}(\bar{\mathbf{M}})$ by Lemma 3.2.1.

LEMMA 3.2.3. *Consider a genus $\bar{\mathbf{M}}$ as above. Then it contains precisely one type in a finite case if and only if $\mathfrak{G}(\bar{\mathbf{M}}) = P$* (cf. Theorem VIII).

PROOF. Sufficiency: Immediate by δ) in Lemma 3.1.6.

Necessity: Assume $\mathfrak{G}(\bar{\mathbf{M}}) \neq P$. Consider an $\bar{\mathbf{M}}$ of this genus in a finite case. If we find a θ not $\in \mathfrak{G}(\bar{\mathbf{M}})$ which is in the range of Theorem VII, then $\bar{\mathbf{M}}^{\theta}$ has the same genus, is also in a finite case and yet $\neq \bar{\mathbf{M}}$, thus completing the proof.

Now in case (I), $\mathfrak{G}(\bar{\mathbf{M}}) = (1)$ by Lemma 2.10.4. Hence any $\theta = 2, 3, \cdots$ will do. And in case (II), P is the range of Theorem VII, so $\mathfrak{G}(\bar{\mathbf{M}}) \neq P$ guarantees the existence of the desired θ.

§3.3 The moment has now come when it is opportune to introduce the notion of a spatial type (cf. footnote [12]). We shall call the abstraction of $\mathbf{M}$ with respect to spatial isomorphism its spatial type. Thus two rings $\mathbf{M}_1$ and $\mathbf{M}_2$ in two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, will be said to have the same spatial type if and only if they are spatially isomorphic. We denote the spatial type of the ring $\mathbf{M}$ by $\ddot{\mathbf{M}}$.

We shall now answer Question II in §2 in the introduction, for the cases (I) and (II)[18], i.e. determine when an algebraic isomorphism determines a spatial one. Thus we shall establish the relationship of $\ddot{\mathbf{M}}$ to $\bar{\mathbf{M}}$. For the cases (I) and (II) we shall determine all spatial types $\ddot{\mathbf{M}}$ in terms of the algebraical types $\bar{\mathbf{M}}$.

The notions $\bar{\mathbf{M}}^{\theta}$, $\mathfrak{G}(\bar{\mathbf{M}})$ as well as the genus $\bar{\mathbf{M}}$ will be basic in these considerations.

A spatial isomorphism of $\mathbf{M}_1$ in $\mathfrak{H}_1$ and $\mathbf{M}_2$ in $\mathfrak{H}_2$ (cf. the Definition A in §2 in the Introduction) will also carry $\mathbf{M}_1'$ into $\mathbf{M}_2'$. So the spatial type $\ddot{\mathbf{M}}$ of a ring $\mathbf{M}$ determines not only the algebraical type $\bar{\mathbf{M}}$ but also the algebraical type $\overline{\mathbf{M}'}$. Now $\bar{\mathbf{M}}$ does not determine $\overline{\mathbf{M}'}$ uniquely, not even for a factor $\bar{\mathbf{M}}$[19]. Consequently a discussion of the spatial type $\ddot{\mathbf{M}}$ of a factor $\mathbf{M}$ must begin with an investigation of the relation between $\bar{\mathbf{M}}$ and $\overline{\mathbf{M}'}$. At this point the notion of genus comes in.

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[18] In the cases (I) of course the answer is known. Nevertheless we include them for the reason given in footnote[1].

[19] The direct factors described in [5], p. 139 or p. 173, Lemma 8.6.1, give elementary examples of this. The exhaustive results in this respect are contained in the last part of our Theorem X.

LEMMA 3.3.1. *For every factor* $\mathbf{M}$ *in the cases* (I) *and* (II)

$$\overline{\overline{\mathbf{M}'} = \overline{\mathbf{M}}^c}.$$

PROOF. Due to our assumptions concerning $\mathbf{M}$ there exists a closed linear set $\mathfrak{M} \eta \mathbf{M}$ which is not $= (0)$ and has a finite $D_{\mathbf{M}}(\mathfrak{M})$. Choose an $f \epsilon \mathfrak{M}$ with $f \neq 0$ and form $\mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{M}_f^{\mathbf{M}}$ by [5], p. 143, Def. 5.1.1. These are closed linear sets, clearly $\eta \mathbf{M}$ and $\eta \mathbf{M}'$ respectively, both $\neq (0)$ and $\mathfrak{M}_f^{\mathbf{M}'} \subseteq \mathfrak{M}$ since $f \epsilon \mathfrak{M} \eta \mathbf{M}$. Hence $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ is finite.

We now normalize $D_{\mathbf{M}}$ so as to make $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = 1$ and then $D_{\mathbf{M}'}$ so as to make $C = 1$ for the C of [5], p. 182, Theorem X. Hence

$$D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}}) = 1. \tag{3.3.α}$$

For the sake of brevity we write

$$\mathfrak{M}_1 = \mathfrak{M}_f^{\mathbf{M}'}, \qquad \mathfrak{M}_1' = \mathfrak{M}_f^{\mathbf{M}}. \tag{3.3.β}$$

Let us now use the considerations of [5], pp. 188–190, §11.4. We form in accord with them, $\mathbf{M}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$, $\mathbf{M}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$. These are coupled factors in $\mathfrak{M}_1 \cdot \mathfrak{M}_1' \neq (0)$ just as $\mathbf{M}$, $\mathbf{M}'$ are in $\mathfrak{H}$. Cf. the remarks, loc. cit., at the beginning of §11.4, and immediately preceding Lemma 11.4.3. We can use $D_{\mathbf{M}}$ and $D_{\mathbf{M}'}$ to obtain relative dimension functions in $\mathbf{M}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$, $\mathbf{M}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$ respectively, as described in Lemma 11.4.2, loc. cit. Our equations (3.3.α) and (3.3.β) are now seen to imply:

First, the relative dimension of $\mathfrak{M}_1 \cdot \mathfrak{M}_1'$ is 1 for $\mathbf{M}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$ and for $\mathbf{M}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$. Hence both factors are in a finite case. Besides $C = 1$ (cf. above). Hence they are either both in a case (I_n), $n = 1, 2, \cdots$ with $1/n$ times the standard normalization, or both in case (II_1) with the standard normalization.[20]

Second, they fulfill the requirements of [6], p. 235, beginning of §4.1, on which the Theorem VI on p. 239 eod. is based.[21] Now this theorem states that the coupled factors to which it applies are dual isomorphic.[22] Again, by [5], p. 188, Lemma 11.4.1, $\mathbf{M}_{(\mathfrak{M}_1)}$, $\mathbf{M}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$ are algebraically ring isomorphic and similarly $\mathbf{M}'_{(\mathfrak{M}_1')}$, $\mathbf{M}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$. Thus $\mathbf{M}_{(\mathfrak{M}_1)}$, $\mathbf{M}'_{(\mathfrak{M}_1')}$ are dual isomorphic, i.e.

$$\overline{\mathbf{M}'_{(\mathfrak{M}_1')}} = \overline{\mathbf{M}_{(\mathfrak{M}_1)}}^c. \tag{3.3.γ}$$

Consequently,

$$\overline{\overline{\mathbf{M}'} = \overline{\mathbf{M}'_{(\mathfrak{M}_1')}} = \overline{\mathbf{M}_{(\mathfrak{M}_1)}}^c = \overline{\mathbf{M}}^c}$$

as desired.

This lemma determines the direction of further analysis of our present topic. It is easy to obtain now more detailed information.

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[20] This discrepancy is of course due to the different ways in which we defined the standard normalization in the cases (I) and (II) (cf. [5], p. 172, Theorem VIII).

[21] The theorem referred to is stated for the case (II_1) only. It holds however for the cases (I_n), $n = 1, 2, \cdots$ too, and with precisely the same proof. Our remark in footnote[1] could have been applied to [6] also.

[22] Called anti-isomorphic, loc. cit.

For this purpose we define

DEFINITION 3.3.1. *For every factor* $\mathbf{M}$ *in the cases* (I) *and* (II) *we form the number*

$$\theta = \frac{1}{C}\frac{D_{\mathbf{M}'}(\mathfrak{H})}{D_{\mathbf{M}}(\mathfrak{H})} \tag{3.3.δ}$$

with any normalization of $D_{\mathbf{M}}$, $D_{\mathbf{M}'}$ *and the corresponding* C *of* [5], *p.* 182, *Theorem* X.

The expression for θ is clearly independent of the normalization of $D_{\mathbf{M}}$, $D_{\mathbf{M}'}$.

Since $0 < C < \infty$, $0 < D_{\mathbf{M}}(\mathfrak{H}), D_{\mathbf{M}'}(\mathfrak{H}) \leqq \infty$, (3.3.$\delta$) characterizes θ as a well defined number with

$$0 \leqq \theta \leqq \infty \tag{3.3.ϵ}$$

except when $D_{\mathbf{M}}(\mathfrak{H}) = D_{\mathbf{M}'}(\mathfrak{H}) = \infty$ i.e. when $\mathbf{M}$, $\mathbf{M}'$ both belong to infinite cases. If this happens, we construe (3.3.δ) to mean

$$\theta = \frac{\infty}{\infty}, \tag{3.3.ζ}$$

the symbol ∞/∞ being considered as an entity different from all numbers of (3.3.ϵ).

LEMMA 3.3.2. *Let* $\mathbf{M}$ *and* θ *be as above. Then we have*

α) *If* $\mathbf{M}$, $\mathbf{M}'$ *are both in infinite cases, then*

$$\overline{\mathbf{M}'} = \overline{\mathbf{M}}^c \quad \text{and} \quad \theta = \frac{\infty}{\infty}.$$

β) *If* $\mathbf{M}$ *is in an infinite case, and* $\mathbf{M}'$ *is in a finite case, then*

$$\overline{\mathbf{M}} = (\overline{\mathbf{M}'^c})^\infty \quad \textit{and} \quad \theta = 0 \quad \textit{so} \quad 1/\theta = \infty.$$

γ) *If* $\mathbf{M}$ *is in a finite case and* $\mathbf{M}'$ *is in an infinite case, then*

$$\overline{\mathbf{M}'} = (\overline{\mathbf{M}}^c)^\infty \quad \textit{and} \quad \theta = \infty.$$

δ) *If* $\mathbf{M}$, $\mathbf{M}'$ *are both in finite cases, then*

$$\overline{\mathbf{M}'} = (\overline{\mathbf{M}}^c)^\theta \quad \textit{or equivalently} \quad \overline{\mathbf{M}} = (\overline{\mathbf{M}'^c})^{\frac{1}{\theta}} \quad \textit{and} \quad 0 < \theta < \infty.$$

PROOF. All assertions concerning θ are immediately by Def. 3.3.1. Let us now consider the other statements.

Ad α), β), γ). Lemma 3.3.1 states that $\overline{\mathbf{M}'}$ and $\overline{\mathbf{M}}^c$ are commensurable. Hence our α), β), γ) follow from α), β), γ) in Lemma 3.1.6 respectively.

Ad δ). The equivalence of these two equations is obvious. We consider the first one.

We choose the normalization of $D_{\mathbf{M}}$, $D_{\mathbf{M}'}$ exactly as in the proof of Lemma 3.3.1. Now (3.3.α), (3.3.β) in that proof give

$$\overline{\mathbf{M}_{(\mathfrak{M}_1)}} = \overline{\mathbf{M}}^{\frac{1}{D_{\mathbf{M}}(\mathfrak{H})}}, \overline{\mathbf{M}'_{(\mathfrak{M}'_1)}} = (\overline{\mathbf{M}'})^{\frac{1}{D_{\mathbf{M}'}(\mathfrak{H})}}$$

Then (3.3.γ) and (2.9.α) yield

$$(\overline{\mathbf{M}'})^{\frac{1}{D_{\mathbf{M}'}(\mathfrak{H})}} = \left(\overline{\mathbf{M}}^{\frac{1}{D_{\mathbf{M}}(\mathfrak{H})}}\right)^c = (\overline{\mathbf{M}}^c)^{\frac{1}{D_{\mathbf{M}}(\mathfrak{H})}}$$

or

$$\overline{\mathbf{M}'} = (\overline{\mathbf{M}}^c)^{\frac{D_{\mathbf{M}'}(\mathfrak{H})}{D_{\mathbf{M}}(\mathfrak{H})}} = (\overline{\mathbf{M}}^c)^{\theta}$$

as desired.

REMARK. The first formula of δ) holds clearly for γ) as well, while it becomes meaningless for α), β). The second formula of δ) holds clearly for β) as well, while it becomes meaningless for α), γ).

It is furthermore apparent from our proof that for δ) the exponents θ, $1/\theta$ are in the range of Theorem VII.

LEMMA 3.3.3. *Let* $\mathbf{M}$ *and* θ *be as above. Then the spatial type* $\ddot{\mathbf{M}}$ *is uniquely determined by the algebraical types,* $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ *and the number* θ.

PROOF. As in the introduction, §2, for each $i = 1, 2$ a Hilbert space $\mathfrak{H}_i$ and an operator ring $\mathbf{M}_i$ in $\mathfrak{H}_i$ must be given. We assume that both $\mathbf{M}_i$ are factors in case (I) or (II) and form for each $\mathbf{M}_i$ its θ_i in the sense of Def. 3.3.1. We assume further that

$$\overline{\mathbf{M}}_1 = \overline{\mathbf{M}}_2, \qquad \overline{\mathbf{M}_1'} = \overline{\mathbf{M}_2'}, \qquad \theta_1 = \theta_2 = \theta.$$

Then our task is to establish $\ddot{\mathbf{M}}_1 = \ddot{\mathbf{M}}_2$ i.e. to find an isomorphic mapping of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ (cf. A) in the Introduction, §2) which carries $\mathbf{M}_1$ into $\mathbf{M}_2$.

In proving this we must distinguish several alternatives corresponding to various values of θ (the joint value of θ_i).

Observe first that the isomorphic mapping of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ which carries $\mathbf{M}_1$ into $\mathbf{M}_2$ also carries $\mathbf{M}_1'$ into $\mathbf{M}_2'$. Thus we can always replace each $\mathfrak{M}_i$ by its $\mathbf{M}_i'$. This has the consequence that $\theta_1 = \theta_2 = \theta$ is replaced by $1/\theta_1 = 1/\theta_2 = 1/\theta$.

Let us now consider the alternatives for θ.

First alternative $0 < \theta < \infty$. Since we may replace θ by $1/\theta$ we can even assume that $0 < \theta \leqq 1$. For both $i = 1, 2$ the factors $\mathbf{M}_i$, $\mathbf{M}_i'$ are in finite cases. Choose the normalizations of both $D_{\mathbf{M}_i}$, $D_{\mathbf{M}_i'}$ so that $D_{\mathbf{M}_i'}(\mathfrak{H}_i) = 1$, $C_i = 1$ (we use the notation of Def. 3.3.1 for both $i = 1, 2$). So $D_{\mathbf{M}_i}(\mathfrak{H}_i) = 1/\theta_i = 1/\theta \geqq 1$. Hence 1 belongs to the range of $D_{\mathbf{M}_i}$ (cf. [5], p. 182, the discussion of the ranges of Δ, Δ_0, Δ', Δ_0' in Theorem X and preceding it). Choose accordingly a $\mathfrak{R}_i \eta \mathbf{M}_i$ with $D_{\mathbf{M}_i}(\mathfrak{R}_i) = 1$. There exists furthermore an ϵ in the range of $D_{\mathbf{M}_1}$ for which there is a $p = 1, 2, \cdots$ with $0 < \epsilon \leqq 1$, $p\epsilon = 1/\theta$.[23] Choose accordingly an $\mathfrak{L}_1 \eta \mathbf{M}_1$ with $D_{\mathbf{M}_1}(\mathfrak{L}_1) = \epsilon$, $\mathfrak{L}_1 \subseteq \mathfrak{R}_1$.

[23] The discussion of [5], p. 182, shows that the ranges of $D_{\mathbf{M}_1}$ and $D_{\mathbf{M}_1'}$ agree for the area $\leqq$ both their maxima, i.e. 1, $1/\theta$. Hence we can argue as follows:

Case (I_n), $n = 1, 2, \cdots$: The range of $D_{\mathbf{M}_1}$ is $(0, 1/n, \cdots, 1)$ now $1/\theta \geqq 1/n$ and $1/n$ is the smallest positive element of the range of $D_{\mathbf{M}_1'}$. Hence that range contains only integer multiples of $1/n$. So $1/\theta = p/n$, $p = 1, 2, \cdots$. So $\epsilon = 1/n$ meets our requirements.

Case (II_1). Choose $p = 1, 2, \cdots$ so that $1/p\theta \leqq 1$. Then $\epsilon = 1/p\theta$ fulfills $p\epsilon = 1/\theta$ and besides $\epsilon \leqq 1, 1/\theta$. Since ϵ belongs to the range of $D_{\mathbf{M}_1}$ it belongs to the range of $D_{\mathbf{M}_1'}$ also.

Let us now use the considerations of [5], pp. 188–190, §11.4. We form, in accord with them, $\mathbf{M}_{i(\mathfrak{R}_i)}$, $\mathbf{M}'_{i(\mathfrak{R}_i)}$. (We choose $\mathfrak{H}_i$, $\mathfrak{R}_i$, $\mathfrak{H}_i$ and $\mathbf{M}_i$, $\mathbf{M}'_i$ for the $\mathfrak{H}$, $\mathfrak{M}$, $\mathfrak{M}'$ and $\mathbf{M}_1$ $\mathbf{M}'$, loc. cit.) These are coupled factors in $\mathfrak{R}_i \neq (0)$ just as $\mathbf{M}_i$, $\mathbf{M}'_i$ are in $\mathfrak{H}_i$. We have

$$\overline{\mathbf{M}'_{i(\mathfrak{R}_i)}} = \overline{\mathbf{M}'_i}.$$

Hence $\overline{\mathbf{M}'_1} = \overline{\mathbf{M}'_2}$ gives

$$\overline{\mathbf{M}'_{1(\mathfrak{R}_1)}} = \overline{\mathbf{M}'_{1(\mathfrak{R}_2)}}.$$

Besides

$$D_{\mathbf{M}_{i(\mathfrak{R}_i)}}(\mathfrak{R}_i) = D_{\mathbf{M}_i}(\mathfrak{R}_i) = 1,$$

$$D_{\mathbf{M}'_{i(\mathfrak{R}_i)}}(\mathfrak{R}_i) = D_{\mathbf{M}'_i}(\mathfrak{H}_i) = 1,$$

and $C_i = 1$ as before. Hence [6], p. 244, Theorem XI, applies with our $\mathfrak{R}_i$, $\mathbf{M}'_{i(\mathfrak{R}_i)}$, $\mathbf{M}_{i(\mathfrak{R}_i)}$ in place of its $\mathfrak{H}_i$, $\mathbf{M}_i$, $\mathbf{M}'_i$. There exists an isomorphic mapping $\mathfrak{J}$ of $\mathfrak{R}_1$ on $\mathfrak{R}_2$ which carries $\mathbf{M}_{1(\mathfrak{R}_1)}$ into $\mathbf{M}_{2(\mathfrak{R}_2)}$ and $\mathbf{M}'_{1,(\mathfrak{R}_1)}$ into $\mathbf{M}'_{2(\mathfrak{R}_2)}$, $\mathfrak{L}_1 \eta \mathbf{M}_1$, $\mathfrak{L}_1 \subseteq \mathfrak{R}_1$ imply $\mathfrak{L}_1 \eta \mathbf{M}_{1(\mathfrak{R}_1)}$. So $\mathfrak{J}$ carries $\mathfrak{L}_1$ into an $\mathfrak{L}_2 \eta \mathbf{M}_{2(\mathfrak{R}_2)}$. It follows that $\mathfrak{L}_2 \eta \mathbf{M}_2$ and $\mathfrak{L}_2 \subseteq \mathfrak{R}_2$. Now

$$D_{\mathbf{M}_2}(\mathfrak{L}_2) = D_{\mathbf{M}_{2(\mathfrak{R}_2)}}(\mathfrak{L}_2) = D_{\mathbf{M}_{1(\mathfrak{R}_1)}}(\mathfrak{L}_1) = D_{\mathbf{M}_1}(\mathfrak{L}_1) = \epsilon.$$

$\mathfrak{J}$ carries $\mathbf{M}_{1(\mathfrak{R}_1)(\mathfrak{L}_1)} = \mathbf{M}_{1(\mathfrak{L}_1)}$ into $\mathbf{M}_{2(\mathfrak{R}_2)(\mathfrak{L}_2)} = \mathbf{M}_{2(\mathfrak{L}_2)}$. Restrict $\mathfrak{J}$ from $\mathfrak{R}_1$ to $\mathfrak{L}_1$ and denote this restricted mapping by $\mathfrak{F}$. Then $\mathfrak{F}$ is an isomorphic mapping of $\mathfrak{L}_1$ on $\mathfrak{L}_2$ which carries $\mathbf{M}_{1(\mathfrak{L}_1)}$ into $\mathbf{M}_{2(\mathfrak{L}_2)}$.

We have $p\epsilon = 1/\theta$ and therefore for both $i = 1, 2$

$$pD_{\mathbf{M}_i}(\mathfrak{L}_i) = D_{\mathbf{M}_i}(\mathfrak{H}_i)$$

or if we introduce the projection $E_i = P_{\mathfrak{L}_i}$ then

$$D_{\mathbf{M}_i}(E_i) = (1/p)D_{\mathbf{M}_i}(1).$$

Consequently we can proceed as in the proof of Lemma 2.6.2 or 2.6.3 to obtain a set of p-order matrix units $W_{i;u,v}$, $u, v = 1, \cdots, p$ with $W_{i;1,1} = E_i$, $\sum_{u=1}^{p} W_{i;u,u} = 1$. Let the projection $F_{i;u} = W_{i;u,u}$ have range $\mathfrak{N}_{i;u}$. We note that $\mathfrak{N}_{i;1} = \mathfrak{L}_i$ and that $W_{i;u,v}$ is partially isometric with initial set $\mathfrak{N}_{i;v}$ and final set $\mathfrak{N}_{i;u}$; i.e. it is an isomorphic mapping of $\mathfrak{N}_{i;v}$ on $\mathfrak{N}_{i;u}$.

Thus $W_{2;u,1}\mathfrak{F}W_{1;1,u} = \mathfrak{F}_u$ is an isomorphic mapping of $\mathfrak{N}_{1;u}$ on $\mathfrak{N}_{2;u}$ for $u = 1, 2, \cdots, p$. For $u = 1$, $W_{i;1,1} = F_{i,1}$, $E_i = P_{\mathfrak{L}_i}$, $\mathfrak{N}_{i;1} = \mathfrak{L}_i$ and hence $\mathfrak{F}_1$ coincides with $\mathfrak{F}$. Since the $\mathfrak{N}_{i;u}$, $u = 1, \cdots, p$ are pairwise orthogonal and span together the closed linear set $\mathfrak{H}_i$ we can combine the $\mathfrak{F}_u$, $u = 1, \cdots, p$ to one isomorphic mapping $\mathfrak{F}^*$ of $\mathfrak{H}_1$ on $\mathfrak{H}_2$. On $\mathfrak{L}_1$ this $\mathfrak{F}^*$ coincides with $\mathfrak{F}_1 = \mathfrak{F}$. Hence

(3.3.η) $\mathfrak{F}^*$ carries $\mathfrak{L}_1$ into $\mathfrak{L}_2$ and in it $\mathbf{M}_{1(\mathfrak{L}_1)}$ into $\mathbf{M}_{2(\mathfrak{L}_2)}$.

By virtue of the properties of $W_{i;u,v}$ (cf. Def. 2.6.1), and owing to the definition of $\mathfrak{F}^*$ we have further

$$(3.3.\theta) \qquad \mathfrak{F}^* \text{ carries } W_{1;u,v} \text{ into } W_{2;u,v}, \qquad u, v = 1, \cdots, p$$

Now $\mathbf{M}_i$ can be characterized in terms of $\mathbf{M}_{i(\mathfrak{L}_1)}$, $\mathfrak{L}_i$ and the $W_{i;u,v}$, $u, v = 1, \cdots, p$. Indeed: $A_i \epsilon \mathbf{M}_i$ is equivalent to $(W_{i;1,v} A_i W_{i;u,1})_{(\mathfrak{L}_i)} \epsilon \mathbf{M}_{i(\mathfrak{L}_i)}$ for all $u, v = 1, \cdots, p$.[24] Consequently (3.3.η) and (3.3.θ) imply that $\mathfrak{F}^*$ carries $\mathbf{M}_1$ into $\mathbf{M}_2$.

This completes the proof of the first alternative.

Second alternative: $\theta = 0$ or ∞. Since we may replace θ by $1/\theta$ it is clear that we need only consider the case $\theta = 0$. For both $i = 1, 2$ the factor $\mathbf{M}_i$ is in an infinite case and the factor $\mathbf{M}_1'$ is in a finite case. Choose the normalization of $D_{\mathbf{M}_i}$, $D_{\mathbf{M}_i'}$ so that

$$D_{\mathbf{M}_i'}(\mathfrak{H}_i) = 1, \qquad C_i = 1.$$

We may parallel our discussion of the first alternative however with $\epsilon = 1$, $p = \infty$. We form the $\mathfrak{R}_i$ accordingly. Since $\epsilon = 1$, $\mathfrak{L}_1 = \mathfrak{R}_1$. $\mathfrak{J}$ obtains as loc. cit. $\mathfrak{L}_1 = \mathfrak{R}_1$ yields $\mathfrak{L}_2 = \mathfrak{R}_2$ and $\mathfrak{F} = \mathfrak{J}$.

Let $E_i = P_{\mathfrak{L}_i}$. We proceed to obtain an infinite number of pairwise orthogonal manifolds $\mathfrak{N}_{i,1}, \mathfrak{N}_{i,2}, \cdots$ with $\mathfrak{N}_{i,1} = \mathfrak{L}_i$, $\mathfrak{N}_{i,u} \eta \mathbf{M}_i$, $D_{\mathbf{M}_i}(\mathfrak{N}_{i,u}) = 1$ for $u = 1, 2, \cdots$, and finally $\mathfrak{H}_i = [\mathfrak{N}_{i,1}, \mathfrak{N}_{i,2}, \cdots]$ (cf. [5], p. 155, Lemma 7.1.2). We obtain the $W_{i,u,v}$ as in the second part of the proof of Lemma 2.6.2.

As in the discussion of the preceding alternative, we now obtain $\mathfrak{F}_u$, $u = 1, 2, \cdots$ and finally $\mathfrak{F}^*$, and by a literal repetition of that argument, the isomorphic mapping $\mathfrak{F}^*$ of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ is seen to carry $\mathbf{M}_1$ into $\mathbf{M}_2$.[25]

Thus the second alternative too is settled.

Third alternative: $\theta = \infty/\infty$. For both $i = 1, 2$ the factors $\mathbf{M}_i$, $\mathbf{M}_i'$ are in infinite cases. Choose a finite $\mathfrak{R}_1 \eta \mathbf{M}_1$, $\mathfrak{R}_1 \neq 0$ and put $P_{\mathfrak{R}_1} = E_1 \epsilon \mathbf{M}$. Under the (algebraic ring) isomorphism of $\mathbf{M}_1$, $\mathbf{M}_2$ (due to $\overline{\mathbf{M}}_1 = \overline{\mathbf{M}}_2$) this projection $E_1 \epsilon \mathbf{M}_1$ corresponds to a projection $E_2 \epsilon \mathbf{M}_2$. Let $\mathfrak{R}_2$ be the range of E_2. Then $\mathfrak{R}_2 \eta \mathbf{M}_2$, $\mathfrak{R}_2 \neq (0)$. Then the isomorphism of $\mathbf{M}_1$, $\mathbf{M}_2$ also induces one of $\mathbf{M}_{1(\mathfrak{R}_1)}$, $\mathbf{M}_{2(\mathfrak{R}_2)}$ owing to [5], p. 187, (i), in Lemma 11.3.3. Thus

$$(3.3.\iota) \qquad \overline{\mathbf{M}_{1(\mathfrak{R}_1)}} = \overline{\mathbf{M}_{2(\mathfrak{R}_2)}}.$$

Also by (ii) eod., $\overline{\mathbf{M}_{i(\mathfrak{R}_i)}'} = \overline{\mathbf{M}_i'}$ hence $\overline{\mathbf{M}_1'} = \overline{\mathbf{M}_2'}$ gives

$$(3.3.\kappa) \qquad \overline{\mathbf{M}_{1(\mathfrak{R}_1)}'} = \overline{\mathbf{M}_{2(\mathfrak{R}_2)}'}.$$

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[24] Put $A_{i,u,v} = W_{i;1,v} A_i W_{i;u,1}$. The forward implication is obvious; the reverse follows from the formula

$$A = \sum_{u,v=1}^{p} W_{i;v,1} A_{i,u,v} W_{i;1,u},$$

which is easily verified.

[25] Footnote[24] applies to this case too. The convergence of $\sum_{u,v=1}^{\infty}$ in footnote[24] is readily seen to be a consequence of the convergence of the sum $\sum_{u=1}^{\infty} W_{i;u,u}$. And in connection with the latter convergence, the remarks made in the proof of (iii) in Lemma 2.6.1 apply again.

Now choose the normalization of $D_{\mathbf{M}_i}$, $D_{\mathbf{M}_i'}$ so that $D_{\mathbf{M}_i}(\mathfrak{R}_i) = 1$, $C_i = 1$. Automatically $D_{\mathbf{M}_i'}(\mathfrak{H}_i) = \infty$. We may then write

$$D_{\mathbf{M}_{i(\mathfrak{R}_i)}}(\mathfrak{R}_i) = D_{\mathbf{M}_i}(\mathfrak{R}_i) = 1$$

$$D_{\mathbf{M}'_{i(\mathfrak{R}_i)}}(\mathfrak{R}_i) = D_{\mathbf{M}_i'}(\mathfrak{H}_i) = \infty$$

and $C_i = 1$. Apply Def. 3.3.1 to $\mathfrak{R}_i$, $\mathbf{M}_{i(\mathfrak{R}_i)}$, $\mathbf{M}'_{i(\mathfrak{R}_i)}$ in place of $\mathfrak{H}$, $\mathbf{M}$, $\mathbf{M}'$ and denote the resulting number by θ_i^0. The above equations give

$$\theta_1^0 = \theta_2^0 = \infty. \tag{3.3.λ}$$

(3.3.ι), (3.3.κ) and (3.3.λ) permit us to apply the second alternative, which we have already disposed of, to $\mathfrak{R}_i$, $\mathbf{M}_{i(\mathfrak{R}_i)}$, $\mathbf{M}'_{i(\mathfrak{R}_i)}$. There exists an isomorphic mapping $\mathfrak{J}$ of $\mathfrak{R}_1$ on $\mathfrak{R}_2$ which carries $\mathbf{M}_{1(\mathfrak{R}_1)}$ into $\mathbf{M}_{2(\mathfrak{R}_2)}$ and $\mathbf{M}'_{1(\mathfrak{R}_1)}$ into $\mathbf{M}'_{2(\mathfrak{R}_2)}$.

We can continue from here on exactly as in the discussion of the second alternative, i.e. paralleling the considerations of the first alternative. We have the $\mathfrak{R}_i$ already. Again we put $\epsilon = 1$, $p = \infty$. Since $\epsilon = 1$, $\mathfrak{L}_1 = \mathfrak{R}_1$. We already have $\mathfrak{J}$. $\mathfrak{L}_1 = \mathfrak{R}_1$ gives $\mathfrak{L}_2 = \mathfrak{R}_2$ and $\mathfrak{F} = \mathfrak{J}$. We have $E_i = P_{\mathfrak{R}_i} = P_{\mathfrak{L}_i}$. The $F_{i,u}$, $u = 1, 2, \cdots$ and $W_{i,u,v}$, $u, v = 1, 2, \cdots$ obtain as loc. cit. So do the $\mathfrak{F}_u$, $u = 1, 2, \cdots$ and finally $\mathfrak{F}^*$; and exactly as loc. cit., the isomorphic mapping $\mathfrak{F}^*$ of $\mathfrak{H}_1$ on $\mathfrak{H}_2$ is seen to carry $\mathbf{M}_1$ into $\mathbf{M}_2$.

Thus the third alternative too is settled and the entire proof is completed.

We settle now the question of existence.

LEMMA 3.3.4. *A factor* $\mathbf{M}$ *in the cases* (I) *or* (II) *with given algebraical types* $\overline{\mathbf{M}}$, $\overline{\mathbf{M}}'$ *and given number* θ (*cf. Def.* 3.3.1) *exists if and only if these fulfill the alternative conditions* α)–δ) *in Lemma* 3.3.2.

PROOF. Necessity: This is merely the assertion of Lemma 3.3.2.

Sufficiency: We shall continue to denote the two prescribed algebraical types by $\overline{\mathbf{M}}$, $\overline{\mathbf{M}'}$ although we have not yet identified them as belonging in this way to any coupled factors $\mathbf{M}$, $\mathbf{M}'$. (In fact, this is what we propose to prove.) The relations α)–δ) in Lemma 3.3.2, which we assumed, imply for the genera

$$\overline{\overline{\mathbf{M}'}} = \overline{\mathbf{M}}^c. \tag{3.3.μ}$$

(This cannot be deduced from Lemma 3.3.1, since $\mathbf{M}$, $\mathbf{M}'$ are fictitions, cf. above.) The genus $\overline{\mathbf{M}}$ contains a (unique) infinite type $\overline{\mathbf{R}}$ (cf. Theorem IX)—$\mathbf{R}$ being an actual factor in a Hilbert space $\mathfrak{R}$.

Consider now the factor $\mathbf{R}'$ in $\mathfrak{R}$. We repeat the construction in §2.4, immediately preceding Theorem IV, with our $\mathfrak{R}$, $\mathbf{R}'$ in place of its $\mathfrak{H}$, $\mathbf{M}$ and with its $p = \infty$. Thus using the notations of §2.4, loc. cit., our $\mathbf{R}'$ is a factor in $\mathfrak{R} = \mathfrak{H} = \mathfrak{H}_2$ and with the ($p = \infty$ dimensional) Hilbert space $\mathfrak{H}_1$, we obtain the Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and in it the ring $\mathbf{M}_1$ of all operators $\langle A'_{t,s} \rangle$ with all $A'_{t,s} \epsilon \mathbf{R}'$ (cf. the similar argument before Theorem IV in §2.4). Put $\mathbf{R}_1 = \mathbf{M}_1'$ i.e. $\mathbf{M}_1 = \mathbf{R}_1'$. It is obvious by the usual matrix computations, that $\mathbf{R}_1 = \mathbf{M}_1'$ is the set of all operators $\langle A\delta_{t,s} \rangle$, $A \epsilon \mathbf{R}$. Hence $\overline{\mathbf{R}}_1 = \overline{\mathbf{R}}$. On the other hand, the discussions of §2.4 show that $\overline{\mathbf{R}_1'} = \overline{\mathbf{M}}_1 = \overline{\mathbf{R}'^{\infty}}$. Thus $\overline{\mathbf{R}_1'}$ is in an infinite case (by Lemma 2.7.1).

Now $\bar{\mathbf{R}}_1 = \bar{\mathbf{R}}$ permits us to replace $\mathfrak{R}$, $\mathbf{R}$ by $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{R}_1$. So we have shown: It is permissible to assume that $\overline{\mathbf{R}'}$ is in an infinite case. We do this, but we continue using the original notations $\mathfrak{R}$, $\mathbf{R}$.

By Lemma 3.3.1, $\overline{\overline{\mathbf{R}'}} = \bar{\mathbf{R}}^c$. Now $\bar{\mathbf{R}} = \bar{\mathbf{M}}$; hence $\overline{\overline{\mathbf{R}'}} = \bar{\mathbf{M}}^c$. By (3.3.$\mu$) this means $\overline{\overline{\mathbf{R}'}} = \bar{\mathbf{M}}'$. Since $\bar{\mathbf{R}}$, $\overline{\mathbf{R}'}$ are both infinite, we have (cf. Theorem IX),

(3.3.ν) $\{\bar{\mathbf{R}}, \overline{\mathbf{R}'}$ are the (unique) infinite types of the genera, $\bar{\mathbf{M}}, \overline{\overline{\mathbf{M}'}}$ respectively.

Choose an $f_0 \neq 0$ in $\mathfrak{R}$ and form the closed linear sets $\mathfrak{M}_{f_0}^{\mathbf{R}'} \eta \mathbf{R}$, $\mathfrak{M}_{f_0}^{\mathbf{R}} \eta \mathbf{R}'$. (cf. [5], p. 143, Def. 5.1.1). We may assume that $\mathfrak{M}_{f_0}^{\mathbf{R}'}$ is finite dimensional. (Choose a finite dimensional $\mathfrak{M} \eta \mathbf{R}$, $\mathfrak{M} \neq (0)$ and then $f_0 \epsilon \mathfrak{M}$. Clearly $\mathfrak{M}_{f_0}^{\mathbf{R}'} \subseteq \mathfrak{M}$.) In this case, $\mathfrak{M}_{f_0}^{\mathbf{R}}$ is automatically finite dimensional. (Use [5], p. 182, Theorem X, for this and for the next step.) Hence both $D_{\mathbf{R}}(\mathfrak{M}_{f_0}^{\mathbf{R}'})$, $D_{\mathbf{R}'}(\mathfrak{M}_{f_0}^{\mathbf{R}})$ are >0, $<\infty$ and $C = D_{\mathbf{R}'}(\mathfrak{M}_{f_0}^{\mathbf{R}})/D_{\mathbf{R}}(\mathfrak{M}_{f_0}^{\mathbf{R}'})$. Now normalize $D_{\mathbf{R}}$, $D_{\mathbf{R}'}$ so that $D_{\mathbf{R}}(\mathfrak{M}_{f_0}^{\mathbf{R}'}) = D_{\mathbf{R}'}(\mathfrak{M}_{f_0}^{\mathbf{R}})$. The above formula for C gives

(3.3.o) $$C = 1.$$

Write $\mathfrak{M}_1 = \mathfrak{M}_{f_0}^{\mathbf{R}'}$, $\mathfrak{M}_1' = \mathfrak{M}_{f_0}^{\mathbf{R}}$ then

(3.3.ξ) $$D_{\mathbf{R}}(\mathfrak{M}_1) = D_{\mathbf{R}'}(\mathfrak{M}_1').$$

We shall now again make use of the considerations of [5], pp. 188–190, §11.4. In accord with them we form $\mathbf{R}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$, $\mathbf{R}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$. These are coupled factors in $\mathfrak{M}_1 \cdot \mathfrak{M}_1' \neq (0)$ just as $\mathbf{R}$, $\mathbf{R}'$ are in $\mathfrak{H}$, and (3.3.ξ) gives

$$D_{\mathbf{R}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}(\mathfrak{M}_1 \cdot \mathfrak{M}_1') = D_{\mathbf{R}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}(\mathfrak{M}_1 \cdot \mathfrak{M}_1').$$

This, together with (3.3.o) and a normalization, shows that the requirements of [6], p. 239, Theorem VI, are fulfilled.[21] Hence $R_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$, $R'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}$ are dual isomorphic,[22] i.e.

$$\overline{\mathbf{R}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}} = \overline{\mathbf{R}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}^c$$

Now $\overline{\mathbf{R}_{(\mathfrak{M}_1)}} = \overline{\mathbf{R}_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}$, $\overline{\mathbf{R}'_{(\mathfrak{M}_1')}} = \overline{\mathbf{R}'_{(\mathfrak{M}_1 \cdot \mathfrak{M}_1')}}$
Consequently

(3.3.π) $$\overline{\mathbf{R}'_{(\mathfrak{M}_1')}} = \overline{\mathbf{R}_{(\mathfrak{M}_1)}}^c.$$

Since $\bar{\bar{\mathbf{R}}} = \bar{\mathbf{M}}$, $\bar{\mathbf{R}}$, $\bar{\mathbf{M}}$ are commensurable. $\bar{\mathbf{R}}$ is in an infinite case. Hence $\alpha)$, $\beta)$ in Lemma 3.1.6 and Lemma 3.1.4 give together that $\bar{\mathbf{M}}$ is a divisor of $\bar{\mathbf{R}}$. Hence by Def. 3.1.1,

(3.3.ρ) $$\bar{\mathbf{M}} = \bar{\mathbf{R}}_{(\mathfrak{L})} \qquad \textit{for a suitable } \mathfrak{L} \eta \mathbf{R}.$$

If $\mathfrak{L}$ is of finite relative dimension, we may assume by [5], Theorem X that $\mathfrak{L} = \mathfrak{M}_{f_0}^{\mathbf{R}'}$ for some f_0. Similarly

(3.3.σ) $$\overline{\mathbf{M}'} = \overline{\mathbf{R}'_{(\mathfrak{L}')}} \qquad \textit{for a suitable } \mathfrak{L}' \eta R', \ \mathfrak{L}' = \mathfrak{M}_{g_0}^{\mathbf{R}}, \ \textit{when finite.}$$

From here on, it is again convenient to distinguish several alternatives, corresponding to various values of θ.

First alternative: $0 < \theta < \infty$. We have the case δ) in Lemma 3.3.2. Hence $\overline{\mathbf{M}'} = (\overline{\mathbf{M}}^c)^\theta$ or by (3.3.ρ) and (3.3.σ)

$$\overline{\mathbf{R}'_{(\mathfrak{Q}')}} = (\overline{\mathbf{R}_{(\mathfrak{Q})}}^c)^\theta$$

Now (3.3.π) and (i) in Lemma 3.2.2, gives

$$\overline{\mathbf{R}'_{(\mathfrak{Q}')}} = (\overline{\mathbf{R}_{\mathfrak{Q}}}^c)^\beta$$

where $\beta = D_{\mathbf{R}'}(\mathfrak{Q}')/D_{\mathbf{R}}(\mathfrak{Q})$. Hence (ii) in Lemma 3.2.2 give

$$\alpha = \beta/\theta = (1/\theta)(D_{\mathbf{R}'}(\mathfrak{Q}')/D_{\mathbf{R}}(\mathfrak{Q})) \in \mathfrak{G}(\overline{\mathbf{R}}) = \mathfrak{G}(\overline{\mathbf{R}_{(\mathfrak{Q})}})$$

The interchange of $\overline{\mathbf{M}}$ and $\overline{\mathbf{M}'}$ and with them of θ and $1/\theta$ as well as $\mathbf{R}$, $\mathfrak{Q}$ and $\mathbf{R}'$, $\mathfrak{Q}'$ replaces α by $1/\alpha$. So we may assume $\alpha \leqq 1$. Hence $\alpha \in \mathfrak{G}(\overline{\mathbf{R}_{(\mathfrak{Q})}})$ secures the constructability of $\overline{\mathbf{R}_{(\mathfrak{Q})}}^\alpha$ in the original sense of §2.8. Thus there exists an $\mathfrak{Q}'' \subset \mathfrak{Q}$ with $\mathfrak{Q}'' \eta \mathbf{R}_{(\mathfrak{Q})}$ and

$$\alpha = \frac{D_{\mathbf{R}(\mathfrak{Q}'')}(\mathfrak{Q}'')}{D_{\mathbf{R}(\mathfrak{Q})}(\mathfrak{Q})} = \frac{D_{\mathbf{R}}(\mathfrak{Q}'')}{D_{\mathbf{R}}(\mathfrak{Q})}$$

($\mathfrak{Q}'' \eta \mathbf{R}_{(\mathfrak{Q})}$ implies $\mathfrak{Q}'' \eta \mathbf{R}$.) $\alpha \in \mathfrak{G}(\overline{\mathbf{R}_{(\mathfrak{Q})}})$ gives

$$\overline{\mathbf{R}_{(\mathfrak{Q}'')}} = \overline{\mathbf{R}_{(\mathfrak{Q})}}^\alpha = \overline{\mathbf{R}_{(\mathfrak{Q})}} = \overline{\mathbf{M}}.$$

Thus we may replace in all our considerations $\mathfrak{Q}$ by $\mathfrak{Q}''$. This carries $\alpha = 1$. Writing again $\mathfrak{Q}$ for $\mathfrak{Q}''$, we see: We may assume that $\alpha = 1$ i.e. that

$$\theta = \frac{D_{\mathbf{R}'}(\mathfrak{Q}')}{D_{\mathbf{R}}(\mathfrak{Q})}.$$

Now consider the coupled factors $\mathbf{R}_{(\mathfrak{Q}\cdot\mathfrak{Q}')}$, $\mathbf{R}'_{(\mathfrak{Q}\cdot\mathfrak{Q}')}$ in $\mathfrak{Q}\cdot\mathfrak{Q}' \neq (0)$. Combining our results, we see that we have

$$\overline{\mathbf{R}_{(\mathfrak{Q}\cdot\mathfrak{Q}')}} = \overline{\mathbf{R}_{(\mathfrak{Q})}} = \overline{\mathbf{M}}.$$
$$\overline{\mathbf{R}'_{(\mathfrak{Q}\cdot\mathfrak{Q})}} = \overline{\mathbf{R}'_{(\mathfrak{Q}')}} = \overline{\mathbf{M}'}.$$

Thus the spatial type $\ddot{\mathbf{R}}_{(\mathfrak{Q}\cdot\mathfrak{Q}')}$ meets all our requirements.

Second alternative: $\theta = 0$ or $\theta = \infty$. Since we may interchange $\overline{\mathbf{M}}$ and $\overline{\mathbf{M}'}$ and with them θ and $1/\theta$ we can even assume that $\theta = 0$. Then we have the case β) in Lemma 3.3.2.

Thus $\overline{\mathbf{M}}$ is in an infinite case. Hence (3.3.ν) gives $\overline{\mathbf{R}} = \overline{\mathbf{M}}$. Now consider the coupled factors $\mathbf{R}_{(\mathfrak{Q}_1)}$, $\mathbf{R}'_{(\mathfrak{Q}')}$ in $\mathfrak{Q}' \neq (0)$. Combining our results we see that we have

$$\overline{\mathbf{R}_{(\mathfrak{Q}')}} = \overline{\mathbf{R}} = \overline{\mathbf{M}}.$$
$$\overline{\mathbf{R}'_{(\mathfrak{Q}')}} = \overline{\mathbf{M}'}$$
$$D_{\mathbf{R}_{(\mathfrak{Q}')}}(\mathfrak{Q}) = D_{\mathbf{R}}(\mathfrak{R}) = \infty.$$
$$D_{\mathbf{R}'_{(\mathfrak{Q}')}}(\mathfrak{Q}') = D_{\mathbf{R}'}(\mathfrak{Q}') < \infty.$$

Thus the spatial type $\ddot{\mathbf{R}}_{(\mathfrak{R}')}$ meets all our requirements.

Third alternative: $\theta = \infty/\infty$. We have the case α) in Lemma 3.3.2. Hence $\bar{\mathbf{M}}$, $\overline{\mathbf{M}'}$ are both in infinite cases, and so (3.3.ν) gives

$$\bar{\mathbf{R}} = \bar{\mathbf{M}}$$

$$\overline{\mathbf{R}'} = \overline{\mathbf{M}'}.$$

Also

$$D_{\mathbf{R}}(\mathfrak{R}) = D_{\mathbf{R}'}(\mathfrak{R}) = \infty.$$

Thus the spatial type $\ddot{\mathbf{R}}$ meets all our requirements.

This completes the proof.

Summing up the results of Lemmas 3.3.3 and 3.3.4,

THEOREM X. *Consider factors* $\mathbf{M}$ *in the cases* (I) *and* (II). *Then the spatial type* $\ddot{\mathbf{M}}$ *is uniquely determined by the algebraical types* $\bar{\mathbf{M}}$, $\overline{\mathbf{M}'}$ *and the number* θ (*cf. Def.* 3.3.1).

$\ddot{\mathbf{M}}$ *exists if and only if* $\bar{\mathbf{M}}$, $\overline{\mathbf{M}'}$ *and* θ *fulfill the alternative conditions* $\alpha) \cdots \delta)$ *in Lemma* 3.3.2.

REMARK. In some cases, two of $\bar{\mathbf{M}}$, $\overline{\mathbf{M}'}$ and θ suffice to determine the third one and thus $\ddot{\mathbf{M}}$. Specifically,

α) If $\bar{\mathbf{M}}$ is in a finite case, then $\bar{\mathbf{M}}$ and θ determine $\overline{\mathbf{M}'}$ by γ), δ) in Lemma 3.3.2.

β) If $\overline{\mathbf{M}'}$ is in a finite case, then $\overline{\mathbf{M}'}$ and θ determine $\bar{\mathbf{M}}$ by β), δ) in Lemma 3.3.2.

γ) If either $\bar{\mathbf{M}}$ or $\overline{\mathbf{M}'}$ is in an infinite case, then $\bar{\mathbf{M}}$, $\overline{\mathbf{M}'}$ determine θ by α), β) or γ) in Lemma 3.3.2.

δ) If $\bar{\mathbf{M}}$, $\overline{\mathbf{M}'}$ are both in finite cases, then $\bar{\mathbf{M}}$, $\overline{\mathbf{M}'}$ determine θ precisely up to a factor belonging to the group $\mathfrak{G}(\bar{\mathbf{M}})$ owing to δ) in Lemma 3.3.2 and to (2.10.α) and Lemma 3.2.1. Consequently they determine θ outright if and only if $\mathfrak{G}(\bar{\mathbf{M}}) = (1)$.[26]

Thus the only remaining objects for our inquiries in the cases (I) and (II) are the algebraical types $\bar{\mathbf{M}}$. Theorems IX and X (or Lemmas 3.2.1 and 3.2.2 in place of the latter) show that the knowledge of these is equivalent to the knowledge of their genera $\ddot{\mathbf{M}}$. The case (I) is completely clear; hence the above concepts must be studied in the case (II). And we can analyze a genus $\ddot{\mathbf{M}}$ in case (II) by analyzing any one of its types $\bar{\mathbf{M}}$ in a finite case. Consequently we need only be concerned with (algebraical) types $\bar{\mathbf{M}}$ in case (II_1).

CHAPTER IV. APPROXIMATE FINITENESS

§4.1 In accordance with the above we begin the systematic investigation of the algebraical types $\bar{\mathbf{M}}$ in case (II_1). Throughout this chapter $\mathbf{M}$ will be a factor in case (II_1) in Hilbert space $\mathfrak{H}$. We use $D_{\mathbf{M}}(E)$, (but now in the standard

[26] Hence θ is never needed in the cases (I) by Lemma 2.10.4 The situation is different in the cases (II). Cf. Theorem XV and the remark at the end of §5.6.

normalization) $Tr_{\mathfrak{M}}(A)$, $[[A]]$ as in §1.2. We shall also consider other factors in $\mathfrak{H}$ but they will be denoted by different letters. All our considerations however will take place in $\mathfrak{H}$.

We begin by stating some familiar facts about factors in the case (I_n), $n = 1, 2, \cdots$.

LEMMA 4.1.1. *Let* $\mathbf{N}$ *be a factor in a case* (I_p), $p = 1, 2, \cdots$. *Then there exists a one-to-one correspondence between the two following sets of entities,*

(α) *The (algebraical ring) isomorphisms* $\mathcal{K}$ *between* $\mathbf{N}$ *and the system of all complex p-order matrices*

$$a = \{a_{t,s}\}, \qquad (t, s = 1, \cdots, p)^{27}$$

(β) *The matrix bases of* $\mathbf{N}$, *i.e. the systems* $W_{u,v}$, $(u, v = 1, \cdots, p)$ *of p order matrix units, that are contained in* $\mathbf{N}$ *and have*

$$E_0 = \sum_{u=1}^{p} W_{u,u} = 1. \tag{4.1.α}$$

The correspondence is this:

(i) $\mathcal{K}$ *is given:*

$$W_{u,v} = \mathcal{K}^{-1}\{\delta_{t,v}\delta_{s,u}\}$$

(ii) *The* $W_{u,v}$ *are given:*

$$\mathcal{K}A = a = \{a_{t,s}\}$$

is equivalent to

$$A = \sum_{u=1}^{p} \sum_{v=1}^{p} a_{v,u} W_{u,v}$$

PROOF.[28] (i) leads from (α) to (β). The conditions of Def. 2.6.1 as well as (4.1.α) are verified by direct matrix computation.

(ii) leads from (β) to (α): The rules of matrix computation are verified from Def. 2.6.1 and (4.1.α).

(i) implies (ii): Immediate by matrix computation.

(ii) implies (i). Put $a_{u,v} = \delta_{t,v}\delta_{s,u}$ in (ii).

REMARK 1. It is clear by (ii) that the ring $\mathbf{N}$ is generated by the $W_{u,v}$, $u, v = 1, \cdots, p$,

$$\mathbf{N} = \mathbf{R}(W_{u,v}\,;\, u, v = 1, \cdots, p).$$

REMARK 2. The two sets (α), (β) are of course not empty. This follows in numerous ways from past results; the equivalent was even stated explicitly in the Corollary to Lemma 2.10.3.

LEMMA 4.1.2. *Let* $\mathbf{N}$, $\mathbf{O}$ *be two factors in the cases* (I_p), (I_q) *respectively*, $p, q, = 1, 2, \cdots$. *Assume that*

$$\mathbf{N} \subseteq \mathbf{O}.$$

---

[27] The (complex numerical) matrices which we consider now should be viewed as endowed with the same operations and defined in the same way as the (operator) matrices of B_p) in §2.4.

[28] Given only for the sake of completeness.

Then we have

(i) *p is a divisor of q: $q = pr$, $r = 1, 2, \cdots$.*

(ii) *To every matrix basis, $W_{u,v}$, $u, v = 1, \cdots, p$ of $\mathbf{N}$ there exists a matrix base $X_{o,w}$, $o, w = 1, \cdots, q$ of $\mathbf{O}$ such that*

$$W_{u,v} = \sum_{i=1}^{r} X_{(u-1)r+i,(v-1)r+i}$$

(iii) *If we pass from the matrix bases $W_{u,v}$ and $X_{o,w}$ in* (ii) *to the isomorphism $\mathcal{K}$ and $\mathfrak{L}$ which correspond to them by Lemma* 4.1.1, *then the correlation of* (ii) *is equivalent to this:*

For $A \in \mathbf{N} \subseteq \mathbf{O}$, both $\mathcal{K}A = a = \{a_{t,s}\}$, $t, s = 1, \cdots, p$ and $\mathfrak{L}A = \bar{a} = \{\bar{a}_{k,l}\}$ $k, l = 1, \cdots, q$ are defined and

$$\bar{a}_{k,l} = a_{t,s}\delta_{i,j} \quad \text{for} \quad k = (t-1)r + i, \qquad l = (s-1)r + j.$$

PROOF. Ad (i). $D_{\mathbf{O}}(E)$ will do as $D_{\mathbf{N}}(E)$ for the projections $E \in \mathbf{N} \subseteq \mathbf{O}$ in the sense of [5], p. 165, Def. 8.2.1[29] and standard normalization. And since the dimension is uniquely characterized by those properties, therefore $D_{\mathbf{O}}(E) = D_{\mathbf{N}}(E)$ for these E.

Hence the total range of $D_{\mathbf{O}}(E)$ contains the total range of $D_{\mathbf{N}}(E)$ i.e. the set $(0, 1/q, \cdots, 1)$ contains the set $(0, 1/p, \cdots, 1)$. Consequently, p is a divisor of q.

Ad (ii). Consider a matrix base $W_{u,v}$ of $\mathbf{N}$ and an arbitrary matrix base $X'_{o,w}$ of $\mathbf{O}$. Put

$$W'_{u,v} = \sum_{i=1}^{r} X_{(u-1)r+i,(v-1)r+i}. \tag{4.1.β}$$

Then one verifies by explicit computation (using Def. 2.6.1) that the $W'_{u,v}$ also form a system of p-order matrix units. Also

$$\sum_{u=1}^{p} W'_{u,u} = \sum_{o=1}^{q} X'_{o,o} = 1.$$

Since $D_{\mathbf{O}}(W'_{u,u}) = D_{\mathbf{O}}(W'_{1,1})$ we have $D_{\mathbf{O}}(W'_{1,1}) = 1/p$ and similarly $D_{\mathbf{O}}(W_{1,1}) = 1/p$. Therefore there exists a partially isometric U in $\mathbf{O}$ with the initial projection $W'_{1,1}$ and the final projection $W_{1,1}$. Now put

$$V = \sum_{u=1}^{p} W_{u,1} U W'_{1,u}.$$

Clearly $V \in \mathbf{O}$ and one verifies by direct computation

$$V^*V = VV^* = 1 \qquad \text{i.e. } V \text{ is unitary.} \tag{1}$$

$$VW'_{u,v} = W_{u,v}V \qquad \text{i.e. } W_{u,v} = VW'_{u,v}V^{-1} \tag{2}$$

Hence replacement of $X'_{u,v}$ by $VX'_{u,v}V^{-1} = X_{u,v}$ will leave all its properties intact, but replace $W'_{u,v}$ by $W_{u,v} = VW'_{u,v}V^{-1}$. And now (4.1.$\beta$) yields the desired equation of (ii).

Ad (iii). Apply (ii) in Lemma 4.1.1 to $\mathcal{K}$ (with $W_{u,v}$) and $\mathfrak{L}$ (with $X_{o,w}$). Then the equation of (ii) goes over into the desired relation of (iii).

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[29] Consider also p. 170, Lemma 8.3.5, loc. cit. We are using the fact that $\mathbf{O}$ belongs to a finite case.

We now define

DEFINITION 4.1.1. **M** *is approximately finite of type* $[p_1, p_2, \cdots]$ *if this is true. There exists a sequence of factors* $\mathbf{N}_1, \mathbf{N}_2, \cdots$ *with the following properties:*

(i) $\mathbf{N}_n$ *is in case* (I_{p_n})

(ii) $\mathbf{N}_1 \subset \mathbf{N}_2 \subset \cdots$.

(iii) *The ring* **M** *is generated by the* $\mathbf{N}_1, \mathbf{N}_2, \cdots$.

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n ; n = 1, 2, \cdots)$$

We have immediately:

LEMMA 4.1.3. *Under the conditions of Def.* 4.1.1, *we have necessarily*

(i) p_n *is a divisor of* p_{n+1}, $p_{n+1} = p_n r_n$ $(r_n = 1, 2, \cdots)$.

(ii) $p_n \to \infty$ *for* $n \to \infty$ *i.e. infinitely many* $r_n > 1$.

PROOF. Ad (i). As $\mathbf{N}_n \subseteq \mathbf{N}_{n+1}$ this follows from (i) in Lemma 4.1.2.

Ad (ii). Owing to (i), $p_1 \leqq p_2 \leqq \cdots$ so failure of (ii) implies $p_m = p_{m+1} = p_{m+2} = \cdots$ for some m. Hence equally $\mathbf{N}_m = \mathbf{N}_{m+1} = \mathbf{N}_{m+2} = \cdots$[30] and so (iii) in Def. 4.1.1 gives

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) = \mathbf{N}_m.$$

But this is impossible since **M** is in the case (II_1) and $\mathbf{N}_m$ in the case (I_{p_n}).

For every sequence $[p_1, p_2, \cdots]$ which meets these requirements there exists an **M** which is approximately finite of that type. But the concept introduced by Def. 4.1.1 is a preliminary one and we prefer to prove this existence only in conjunction with the final reformulation of the concept of approximate finiteness in Theorems XII and XIV. We now proceed in a different direction.

LEMMA 4.1.4. *For any sequence of rings* $\mathbf{N}_1, \mathbf{N}_2, \cdots$ *which fulfills conditions* (ii), (iii) *in Def.* 4.1.1, *denote by* **S** *the set theoretical sum of that sequence. Then we have*

(i) **S** *is an algebra.*

(ii) **M** *is the set of elements of* **B** *which are limits of a convergent sequence from* **S**.

PROOF. Ad (i): Every $\mathbf{N}_n$ is a ring; hence an algebra. Also $\mathbf{N}_1 \subset \mathbf{N}_2 \subset \cdots$. Hence the sum **S** is an algebra also.

Ad (ii): Denote the set of limits of all metrically convergent sequences from **S** by $\mathbf{S}_1$. We must prove that $\mathbf{S}_1 = \mathbf{M}$.

Now

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) = \mathbf{R}(\mathbf{S})$$

and $\mathbf{S} \subseteq \mathbf{S}_1 \subseteq \mathbf{M}$. Hence $\mathbf{S}_1 = \mathbf{M}$ is established if we can show that $\mathbf{S}_1$ is a ring.

S is an algebra and αA, A^* (as one-variable functions), $A + B$ (as a two-variable function), and AB (as a one-variable function with respect to either variable) are continuous for metric convergence. (This follows from the evaluations of [6], p. 242, property II°.) Hence $\mathbf{S}_1$ is also an algebra. Therefore

[30] For $l \geqq m$, $\mathbf{N}_m$ and $\mathbf{N}_l$ contain the same number of linearly independent elements: $p_l^2 = p_m^2$. As $\mathbf{N}_m \subseteq \mathbf{N}_l$ this implies $\mathbf{N}_m = \mathbf{N}_l$.

its closure with respect to metric convergence implies its closure in the weak topology by Theorem I. Consequently, $\mathbf{S}_1$ is a ring.

This completes the proof.

We can now prove

THEOREM XI. *The (algebraical) type* $\overline{\mathbf{M}}$ *of an* $\mathbf{M}$ *which is approximately finite of type* $[p_1, p_2, \cdots]$ *depends on the sequence* $[p_1, p_2, \cdots]$ *alone and not on the particular choice of* $\mathbf{M}$.

(Provided such an $\mathfrak{M}$ exists at all. Cf. above.)

PROOF. Consider two such $\mathbf{M}^{(h)}$, $h = 1, 2$, and form their $\mathbf{N}_1^{(h)}, \mathbf{N}_2^{(h)}, \cdots$ by Def. 4.1.1 and their $\mathbf{S}^{(h)}$ by Lemma 4.1.4. We wish to prove that $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ are algebraically isomorphic.

We shall choose by induction for every $n = 1, 2, \cdots$ an (algebraical) isomorphism $\mathfrak{K}_n^{(h)}$ between $\mathbf{N}_n^{(h)}$ and the system of all (complex) p_n-order matrices in the sense of (α) in Lemma 4.1.1. For $n = 1$ we choose a $\mathfrak{K}_1^{(h)}$ which meets this requirement (cf. Remark 2 after Lemma 4.1.1). If for any $n = 2, 3, \cdots$, $\mathfrak{K}_{n-1}^{(h)}$ has already been chosen, then we choose $\mathfrak{K}_n^{(h)}$ in the following way: Apply Lemma 4.1.2 to $\mathbf{N}_{(n-1)}^{(h)}, \mathbf{N}_n^{(h)}, \mathfrak{K}_{n-1}^{(h)}$ in place of its $\mathbf{N}$, $\mathbf{O}$, $\mathfrak{K}$ (with $\mathbf{M}^{(h)}$ in place of $\mathbf{M}$) and put $\mathfrak{K}_n^{(h)} = \mathfrak{L}$. Thus $\mathfrak{K}_{n-1}^{(h)}$ and $\mathfrak{K}_n^{(h)}$ are related to each other as $\mathfrak{K}$ and $\mathfrak{L}$ in (iii) in Lemma 4.1.2.

Now form the (algebraical) isomorphism $\mathfrak{J}_n = (\mathfrak{K}_n^{(2)})^{-1}\mathfrak{K}_n^{(1)}$, of $\mathbf{N}_n^{(1)}$ and $\mathbf{N}_n^{(2)}$. In view of the relationship given in the last sentence of the preceding paragraph, $\mathfrak{J}_n$ is a part of $\mathfrak{J}_{n+1}$. Thus the isomorphisms $\mathfrak{J}_n$ of $\mathbf{N}_n^{(1)}$ and $\mathbf{N}_n^{(2)}$ for $n = 1, 2, \cdots$ are compatible with each other and they merge to an (algebraical) isomorphism $\mathfrak{J}$ of $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$. In $\mathbf{N}_n^{(1)}$, our $\mathfrak{J}$ agrees with $\mathfrak{J}_n$. Hence $\mathfrak{J}$ maps $\mathbf{N}_n^{(1)}$ on $\mathbf{N}_n^{(2)}$.

In $\mathbf{N}_n^{(h)}$, $Tr_{\mathbf{N}_n^{(h)}}(A)$ is purely algebraical (cf. §1.2). Clearly $Tr_{\mathbf{M}^{(h)}}(A)$ will do as $Tr_{\mathbf{N}_n^{(h)}}(A)$ for the $A \in \mathbf{N}_n^{(h)}$ in the sense of [6], p. 219, Property IV. And as the trace is uniquely determined by that property, so $Tr_{\mathbf{M}^{(h)}}(A) = Tr_{\mathbf{N}_n^{(h)}}(A)$ for $A \in \mathbf{N}_n^{(h)}$. Hence $[[A]] = (Tr_{\mathbf{M}^{(h)}}(A^*A))^{\frac{1}{2}} = (Tr_{\mathbf{N}_n^{(h)}}(A^*A))^{\frac{1}{2}}$ for $A \in \mathbf{N}_n^{(h)}$ and therefore $[[A]]$ and with it $[[A - B]]$ is purely algebraical in $\mathbf{N}_n^{(h)}$.

Thus $\mathfrak{J}$ leaves $[[A - B]]$ invariant in $\mathbf{N}_n^{(h)}$ for all $n = 1, 2, \cdots$. Hence the same is true in all $\mathbf{S}^{(h)}$.

Summing up: $\mathfrak{J}$ is an isomorphism of $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ with respect to

$$1, \alpha A, A^*, A + B, AB, \tag{4.1.γ}$$

and

$$[[A - B]]. \tag{4.1.δ}$$

By (4.1.δ) $\mathfrak{J}$ is isometric. Now extend $\mathfrak{J}$ as far as possible in $\mathbf{B}$ by metric convergence, thus obtaining a new isometric (and hence one-to-one) correspondence $\mathfrak{F}$. By (ii) in Lemma 4.1.4, $\mathfrak{F}$ maps all of $\mathbf{M}^{(1)}$ on $\mathbf{M}^{(2)}$.

The continuity of the operations (4.1.γ) (cf. the proof of (ii) in Lemma 4.1.4) guarantees that $\mathfrak{F}$ too leaves (4.1.γ) invariant. Hence $\mathfrak{F}$ is an (algebraical) isomorphism of $\mathbf{M}_1$ and $\mathbf{M}_2$. This completes the proof.

§4.2 As a preliminary before introducing other definitions of approximate finiteness, we develop further the ideas which appear in §1.5.

LEMMA 4.2.1. *For any $\epsilon > 0$ there exists an $w_3 = w_3(\epsilon) > 0$ with the following property:*

Consider a ring $\mathbf{N} \subseteq \mathbf{M}$ and a projection $E \in \mathbf{M}$. If there exists an $A \in \mathbf{N}$ with $[[A - E]] < w_3$ then there exists also a projection $F \in \mathbf{N}$ with $[[F - E]] < \epsilon$.

PROOF. Consider first the hypothetical A. We have $[[A^* - E]] = [[(A - E)^*]] = [[A - E]] < w_3$. Since $\frac{1}{2}(A + A^*) - E = \frac{1}{2}((A - E) + (A^* - E))$ these inequalities imply $[[\frac{1}{2}(A + A^*) - E]] < w_3$ by the triangle inequality. Thus we may replace A by $\frac{1}{2}(A + A^*)$ i.e. we may assume that A is Hermitian.

Now form the following functions

$$\psi_1(\lambda) \begin{cases} = 1 & \text{for } |\lambda| \geqq 1 \\ = 2|\lambda| - 1 & \text{for } \frac{1}{2} \leqq |\lambda| < 1 \\ = 0 & \text{for } |\lambda| \leqq \frac{1}{2}, \end{cases}$$

$$\psi_2(\lambda) \begin{cases} = 1 & \text{for } |\lambda| \geqq \frac{1}{2} \\ = 2|\lambda| & \text{for } |\lambda| < \frac{1}{2}. \end{cases}$$

Clearly Lemma 1.5.3 applies to both of them. We form the $w_2(\epsilon)$ of that lemma for both functions and denote their minimum by $w_2'(\epsilon)$. We put

$$w_3(\epsilon) = w_2'\left(\frac{\epsilon}{2}\right). \tag{4.2.α}$$

Form the further function

$$\varphi(\lambda) \begin{cases} = 1 & \text{for } |\lambda| \geqq \frac{1}{2} \\ = 0 & \text{for } |\lambda| < \frac{1}{2}. \end{cases}$$ [31]

The following facts are clear:

$\varphi(\lambda) = \overline{\varphi(\lambda)} = \varphi(\lambda)^2$. Hence $F = \varphi(A)$ is a projection which belongs to $\mathbf{M}$ along with A. $\psi_1(\lambda) = \psi_2(\lambda) = \varphi(\lambda) = \lambda$ for $\lambda = 0, 1$, the entire spectrum of E. Hence

$$\psi_1(E) = \psi_2(E) = \varphi(E) = E. \tag{4.2.β}$$

Finally $(1 - \varphi(\lambda))\psi_1(\lambda) + \varphi(\lambda)\psi_2(\lambda) = \varphi(\lambda)$. Hence

$$(1 - F)\psi_1(A) + F\psi_2(A) = F. \tag{4.2.γ}$$

Now $[[A - E]] < w_3(\epsilon)$ implies by Lemma 1.5.3 and (4.2.α) that $[[\psi_1(A) - \psi_1(E)]] < \epsilon/2$ and $[[\psi_2(A) - \psi_2(E)]] < \epsilon/2$. Hence (4.2.$\beta$) now yields that $[[\psi_1(A) - E]] < \epsilon/2$ and $[[\psi_2(A) - E]] < \epsilon/2$. Since $|||F|||$ and $|||1 - F|||$ are $\leqq 1$ [32] we can therefore conclude that

$$[[(1 - F)(\psi_1(A) - E) + F(\psi_2(A) - E)]] < \epsilon.$$

[31] Observe that $\psi_1(\lambda)$, $\psi_2(\lambda)$ are continuous, while $\varphi(\lambda)$ is not.

[32] Actually we have an equality in the case of each of these operators except when it is zero.

This may be rewritten

$$[[(1 - F)\psi_1(A) + F\psi_2(A) - E]] < \epsilon.$$

With (4.2.γ) this gives $[[F - E]] < \epsilon$ as desired.

LEMMA 4.2.2. *For any $\epsilon > 0$, there exists an $w_4 = w_4(\epsilon) > 0$ with the following property.*

Consider two projections $E, F \in \mathbf{M}$ with the closed linear sets $\mathfrak{M}, \mathfrak{N}$. If $[[E - F]] < w_4$, then there exists a partially isometric $W' \in \mathbf{M}$ with an initial set $\mathfrak{M}' \subseteq \mathfrak{M}$ a final set $\mathfrak{N}' \subseteq \mathfrak{N}$ and with $[[W' - E]] < \epsilon$.

PROOF. Form the canonical decomposition of $A = FE$ in the sense of [5], p. 143, Def. 4.4.1:

$$(4.2.\delta) \qquad FE = A = W'B. \qquad EF = A^* = BW'^*.$$

Here B is Hermitian and definite, W' is partially isometric with initial set $\mathfrak{M}' =$ [Range A^*] = [Range EF] $\subseteq$ [Range E] = $\mathfrak{M}$ and final set $\mathfrak{N}'$ = [Range A] = [Range FE] $\subseteq$ [Range F] = $\mathfrak{N}$.

The range of W'^* is $\mathfrak{M}' \subset \mathfrak{M}$ = [Range E]. Thus $EW'^* = W'^*$ and taking adjoints, we obtain

$$(4.2.\epsilon) \qquad W'E = W'.$$

For any canonical decomposition $B^2 = A^*A$. In our case $A^*A = EF \cdot FE = EFE$ so

$$(4.2.\zeta) \qquad B^2 = EFE$$

Observe also that due to the character of these operators

$$(4.2.\eta) \qquad |||E||| \leqq 1, \qquad |||F||| \leqq 1, \qquad |||W'||| \leqq 1.^{32}$$

Thus (4.2.ζ) implies $|||B^2||| \leqq 1$ and so

$$(4.2.\theta) \qquad |||B||| \leqq 1.$$

Now form the function

$$\psi(\lambda) \begin{cases} = 1 & \text{for } |\lambda| \geqq 1 \\ = \sqrt{|\lambda|} & \text{for } |\lambda| \leqq 1. \end{cases}$$

Clearly Lemma 1.5.3 applies to it. We form the $w_2(\epsilon)$ of that lemma for this function and put

$$(4.2.\iota) \qquad w_4(\epsilon) = \operatorname{Min}\left(w_2\left(\frac{\epsilon}{2}\right), \frac{\epsilon}{2}\right).$$

It is clear that $\psi(\lambda^2) = \lambda$ for $0 \leqq \lambda \leqq 1$ which includes the entire spectrum of B, since B is definite and (4.2.θ) holds. Thus $\psi(B^2) = B$, or using (4.2.ζ) we obtain

$$(4.2.\kappa) \qquad \psi(EFE) = B.$$

On the other hand, $\psi(\lambda) = \lambda$ for $\lambda = 0, 1$ the spectrum of E and hence

$$\psi(E) = E. \tag{4.2.λ}$$

Now our assumption, $[[E - F]] < w_4$ implies by (4.2.η) that $[[(E - F)E]] < w$ and $[[E(E - F)E]] < w_4$ or

$$[[E - FE]] < w_4\,, \qquad [[E - EFE]] < w_4 \tag{4.2.μ}$$

(4.2.μ), Lemma 1.5.3 and (4.2.ι) together imply

$$[[\psi(E) - \psi(EFE)]] < \frac{\epsilon}{2}.$$

This and (4.2.κ) and (4.2.λ) yield $[[E - B]] < \epsilon/2$. This and (4.2.η) imply $[[W'(E - B)]] < \epsilon/2$. Hence (4.2.$\epsilon$) and (4.2.$\delta$) give

$$[[W' - FE]] < \frac{\epsilon}{2}. \tag{4.2.ν}$$

(4.2.μ) and (4.2.ι) imply $[[E - FE]] < \epsilon/2$. Hence (4.2.ν) yields $[[W' - E]] < \epsilon$ as desired.

LEMMA 4.2.3. *If $U \in \mathbf{M}$ is unitary and $W \in \mathbf{M}$ is partially isometric, and if U agrees with W on its initial set, then*

$$[[U - W]] \leqq (2\,[[W - 1]])^{\frac{1}{2}}$$

PROOF. The initial set of W is the range or final set of W^*. Therefore $UW^* = WW^*$. This implies for the adjoints $WU^* = WW^*$. Hence

$$\begin{aligned}(U - W)(U - W)^* &= UU^* - UW^* - WU^* + WW^* \\ &= 1 - WW^* - WW^* + WW^* = 1 - WW^*.\end{aligned}$$

Therefore $[[U - W]]^2 = Tr_{\mathbf{M}}((U - W)(U - W)^*) = Tr_{\mathbf{M}}(1 - WW^*)$ or

$$[[U - W]] = (Tr_{\mathbf{M}}(1 - WW^*))^{\frac{1}{2}} \tag{4.2.o}$$

We now make use of Schwartz's inequality in the sense of [6], p. 241, Theorem VIII, remembering that $|||\,W\,||| \leqq 1$ and so $[[W]] \leqq 1$. Thus

$$Tr_{\mathbf{M}}(1 - WW^*) = Tr_{\mathbf{M}}(-W(W - 1)^* - (W - 1)1^*) \leqq 2[[W - 1]].$$

This and (4.2.o) yield the desired inequality of the lemma.

LEMMA 4.2.4. *For any $\epsilon > 0$, there exists an $w_5 = w_5(\epsilon) > 0$ with the following property:*

Consider two projections, $E, F \in \mathbf{M}$ with $D_{\mathbf{M}}(E) = D_{\mathbf{M}}(F)$. If $[[E - F]] < w_5$, then there exists a unitary $U \in \mathbf{M}$ with $F = UEU^{-1}$ and with $[[U - 1]] < \epsilon$.

PROOF. Put $w_5(\epsilon) = w_4(\epsilon')$ where $\epsilon' = \epsilon'(\epsilon) > 0$ will be chosen later.

Apply Lemma 4.2.2 to E, F and their closed linear sets $\mathfrak{M}$, $\mathfrak{N}$. It gives a partially isometric $W' \in \mathbf{M}$ with an initial set $\mathfrak{M}' \subseteq \mathfrak{M}$ and a final set $\mathfrak{N}' \subseteq \mathfrak{N}$ and with

$$[[W' - E]] < \epsilon'. \tag{4.2.ξ}$$

In view of the relation of W' to $\mathfrak{M}'$ and $\mathfrak{N}'$, $D_{\mathbf{M}}(\mathfrak{M}') = D_{\mathbf{N}}(\mathfrak{N}')$. By hypothesis, $D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{N}}(\mathfrak{N})$. Hence $D_{\mathbf{M}}(\mathfrak{M} \ominus \mathfrak{M}') = D_{\mathbf{M}}(\mathfrak{N} \ominus \mathfrak{N}')$. Choose a partially isometric $V' \in \mathbf{M}$ with initial set $\mathfrak{M} \ominus \mathfrak{M}'$ and final set $\mathfrak{N} \ominus \mathfrak{N}'$. Then $U' = W' + V' \in \mathbf{M}$ is also partially isometric, with initial set $\mathfrak{M}$, final set $\mathfrak{N}$, and it agrees with W' on the initial set $\mathfrak{M}'$ of W.

Apply all this to $1 - E$, $1 - F$ and $\mathfrak{M}^{\perp}(= \mathfrak{H} \ominus \mathfrak{M})$, $\mathfrak{N}^{\perp}$ in place of E, F and $\mathfrak{M}$, $\mathfrak{N}$. (Clearly $[[(1 - E) - (1 - F)]] = [[E - F]]$). Thus we obtain a partially isometric $W'' \in \mathbf{M}$ with an initial set $\mathfrak{M}'' \subseteq \mathfrak{M}^{\perp}$ a final set $\mathfrak{N}'' \subseteq \mathfrak{N}^{\perp}$ and with

$$[[W'' - (1 - E)]] < \epsilon'. \tag{4.2.π}$$

Furthermore, there exists a partially isometric $U'' \in \mathbf{M}$ with the initial set $\mathfrak{M}^{\perp}$ final set $\mathfrak{N}^{\perp}$ and agreeing with W'' on the initial set $\mathfrak{M}''$ of W''.

Now put

$$W = W' + W'', \qquad U = U' + U''.$$

Then $W, U \in \mathbf{M}$ are partially isometric, with initial sets $[\mathfrak{M}', \mathfrak{M}'']$, $[\mathfrak{M}, \mathfrak{M}^{\perp}] = \mathfrak{H}$ respectively, and with the respective final sets $[\mathfrak{N}', \mathfrak{N}'']$ and $[\mathfrak{N}, \mathfrak{N}^{\perp}] = \mathfrak{H}$. So U is unitary.

U agrees with U' on the initial set $\mathfrak{M}$ of U'. Hence it maps $\mathfrak{M}$ on $\mathfrak{N}$. Therefore

$$F = UEU^{-1}. \tag{4.2.ρ}$$

U agrees with W on the initial set $[\mathfrak{M}', \mathfrak{M}'']$ of W. Therefore by Lemma 4.2.3,

$$[[U - W]] = (2[[W - 1]])^{\frac{1}{2}} \tag{4.2.σ}$$

(4.2.ξ) and (4.2.π) imply

$$[[W - 1]] < 2\epsilon' \tag{4.2.τ}$$

(4.2.σ) and (4.2.τ) give together $[[U - W]] < (4\epsilon')^{\frac{1}{2}}$. This and (4.2.$\tau$) yield

$$[[U - 1]] < (4\epsilon')^{\frac{1}{2}} + 2\epsilon' \tag{4.2.υ}$$

Hence if we choose ϵ' so that $(4\epsilon')^{\frac{1}{2}} + 2\epsilon' \leqq \epsilon$, (4.2.$\upsilon$) yields

$$[[U - 1]] < \epsilon \tag{4.2.φ}$$

In view, therefore, of (4.2.ρ) and (4.2.φ), the w_5 which we have specified satisfies the condition of our lemma.

§4.3 We shall recast the definition of approximate finiteness in a series of steps, i.e. Defs. 4.3.1, 4.5.2 and 4.6.1, below:

DEFINITION 4.3.1. $\mathbf{M}$ *is approximately finite* (A) *if this is true*:

Given any $A_1, \cdots, A_m \in \mathbf{M}$ *and any* $\epsilon > 0$, *there exists an* $n = n_1(A_1, \cdots,$

A_m, ϵ) *such that for every* $q \geqq n$ *there exists a factor* $\mathbf{N} = \mathbf{N}_1(q, A_1, \cdots, A_m, \epsilon)$ *with the following properties*:

(i) $\mathbf{N}$ *is in case* (I_q)

(ii) $\mathbf{N} \subseteq \mathbf{M}$

(iii) *There exist* $B_1, \cdots, B_m \in \mathbf{N}$ *with*

$$[[B_h - A_h]] < \epsilon \quad \text{for} \quad h = 1, \cdots, m.$$

In the present section we are interested in certain consequences of this definition, in particular what other properties of the approximating factor $\mathbf{N}$ may be assumed. It will be assumed therefore in the five lemmas of this section that $\mathbf{M}$ is approximately finite (A).

LEMMA 4.3.1. *Given any* $A_1, \cdots, A_m \in \mathbf{M}$ *any projection* $E \in \mathbf{M}$ *with* $D_{\mathbf{M}}(E) = 1/p$ *for* $p = 1, 2, \cdots$ *and any* $\epsilon > 0$, *there exists an* $n_2 = n_2(A_1, \cdots, A_m, E, \epsilon)$ *such that for every* $q \geqq n_2$ *which is divisible by* p *there exists a factor* $\mathbf{N} = \mathbf{N}_2(q, A_1, \cdots, A_m, E, \epsilon)$ *with the following properties*:

(i) $\mathbf{N}$ *is in case* (I_q)

(ii) $\mathbf{N} \subseteq \mathbf{M}$.

(iii) *There exist* $B_1, \cdots, B_m \in \mathbf{N}$ *with*

$$[[B_h - A_h]] < \epsilon \quad \textit{for} \quad h = 1, \cdots, m.$$

(iv) *There exists a projection* $F \in \mathbf{N}$ *with* $D_{\mathbf{M}}(F) = D_{\mathbf{M}}(E) = 1/p$, *and* $[[F - E]] < \epsilon$.[33]

PROOF. Apply Def. 4.3.1 with $A_1, \cdots, A_m, E$ in place of its $A_1, \cdots, A_m$ and with ϵ' in place of ϵ where $\epsilon' = \epsilon'(\epsilon)$ will be chosen later.

Put $n_2(A_1, \cdots, A_m, E, \epsilon) = n_1(A_1, \cdots, A_m, E, \epsilon')$ and $\mathbf{N}_2(q, A_1, \cdots, A_m, E, \epsilon) = \mathbf{N}_1(q, A_1, \cdots, A_m, E, \epsilon)$. Now consider a $q \geqq n_2$ which is divisible by p and its $\mathbf{N} = \mathbf{N}_2$.

In view of the above definition of $\mathbf{N}_2$, (i) and (ii) of the present lemma follow from (i) and (ii) of Def. 4.3.1. Furthermore, (iii) of Def. 4.3.1 states that

$$[[B_h - A_h]] < \epsilon' \quad \text{for} \quad h = 1, \cdots, m, \tag{4.3.α}$$

and

$$[[B_{m+1} - E]] < \epsilon'. \tag{4.3.β}$$

Put

$$\epsilon' = \text{Min.}\ (w_3(\epsilon''), \epsilon) \tag{4.3.γ}$$

where $\epsilon'' = \epsilon''(\epsilon)$ will be chosen later. Now (4.3.α) and (4.3.γ) imply (iii) in the lemma, so we must only derive (iv) from (4.3.β).

[33] Observe that this is a strengthened form of Def. 4.3.1.

By Lemma 4.2.1, (4.3.β) and (4.3.γ) imply the existence of a projection $F_1 \epsilon \mathbf{N}$ with

$$[[F_1 - E]] < \epsilon''. \tag{4.3.δ}$$

This implies

$$| D_{\mathbf{M}}(F_1) - D_{\mathbf{M}}(E) | < \epsilon''.^{34} \tag{4.3.ϵ}$$

As $\mathbf{N}$ is in case (I_q) and q is divisible by p, there exists a projection $F \epsilon \mathbf{N}$ with

$$D_{\mathbf{M}}(F) = D_{\mathbf{M}}(E) = 1/p, \tag{4.3.ζ}$$

and we can even choose $F_1 \gtreqless F$. This latter implies $[[F_1 - F]] = (| D_{\mathbf{M}}(F_1) - D_{\mathbf{M}}(F) |)^{\frac{1}{2}}$.[35] This equation, (4.3.ϵ) and (4.3.ζ) together imply $[[F_1 - F]] < \epsilon''^{\frac{1}{2}}$. Hence (4.3.$\delta$) now implies

$$[[F - E]] < \epsilon''^{\frac{1}{2}} + \epsilon''. \tag{4.3.η}$$

Thus if we choose $\epsilon'' = \epsilon''(\epsilon) > 0$, so that $(\epsilon'')^{\frac{1}{2}} + \epsilon'' \leqq \epsilon$ we can conclude from (4.3.η) that $[[F - E]] < \epsilon$. In view of (4.3.ζ) and $F \epsilon \mathbf{N}$, our last inequality shows that $\mathbf{N}$ has property (iv) as desired.

LEMMA 4.3.2. *Given any $A_1, \cdots, A_m \epsilon \mathbf{M}$ any projection $E \epsilon \mathbf{M}$ with $D_{\mathbf{M}}(E) = 1/p$ for $p = 1, 2, \cdots$ and any $\epsilon > 0$. Then there exists an $n_3 = n_3(A_1, \cdots, A_m, E, \epsilon)$ such that for every $q \geqq n_3$ which is divisible by p there exists a factor $\mathbf{N} = \mathbf{N}_3(q, A_1, \cdots, A_m, E, \epsilon)$ with the properties* (i), (ii), (iii) *of Lemma* 4.3.1, *and in addition*

(iv′) $E \epsilon \mathbf{N}$.[33]

PROOF. Our hypotheses are identical with those of Lemma 4.3.1, and we apply the latter to $A_1, \cdots, A_m, E$ but with ϵ' in place of ϵ where

$$\epsilon' = \text{Min. } (w_5(\epsilon''), \epsilon'') \tag{4.3.θ}$$

and $\epsilon'' = \epsilon''(\epsilon) > 0$ will be chosen later.

Denote the n_2, $\mathbf{N}_2$ and $B_1, \cdots, B_m, F$ which we obtain in this way by n_2',

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[34] For any two projections $E, F \epsilon \mathbf{M}$

$$| D_{\mathbf{M}}(F) - D_{\mathbf{M}}(E)| \leqq [[F - E]].$$

Proof. We have

$$| D_{\mathbf{M}}(F) - D_{\mathbf{M}}(E)| = | Tr_{\mathbf{M}}(F) - Tr_{\mathbf{M}}(E)| = | Tr_{\mathbf{M}}(F - E)|$$

This is $\leqq [[1]] \cdot [[F - E]]$ (cf. [6], pp. 241, Theorem VIII); hence $\leqq [[F - E]]$ as desired.

[35] For any two projections $E, F \epsilon \mathbf{M}$ with $E \gtreqless F$

$$[[F - E]] = (| D_{\mathbf{M}}(F) - D_{\mathbf{M}}(E)|)^{\frac{1}{2}}$$

Proof. By symmetry we may assume $E \leqq F$. Then $F - E$ is a projection, $D_{\mathbf{M}}(F) - D_{\mathbf{M}}(E) = D_{\mathbf{M}}(F - E)$. Put $G = F - E$. We must prove $[[G]] = (D_{\mathbf{M}}(G))^{\frac{1}{2}}$. Now

$$[[G]]^2 = Tr_{\mathbf{M}}(G^*G) = Tr_{\mathbf{M}}(G) = D_{\mathbf{M}}(G).$$

$\mathbf{N}_2^+$ and $B_1', \cdots, B_m', F'$. We put $n_3 = n_2'$ and $\mathbf{N}^+ = \mathbf{N}_2^+$. In view of (4.3.$\theta$), Lemma 4.3.1 (iii) gives

$$[[B_h' - A_h]] < \epsilon'' \quad \text{for} \quad h = 1, \cdots, m, \tag{4.3.ι}$$

while (iv) of that Lemma yields

$$[[F' - E]] < w_5(\epsilon'') \tag{4.3.κ}$$

and

$$D_{\mathbf{M}}(F') = D_{\mathbf{M}}(E). \tag{4.3.λ}$$

(4.3.κ) and (4.3.λ) allow us to apply Lemma 4.2.4 with F', E in place of its E, F. Consequently there exists a unitary $U \in \mathbf{M}$ with

$$[[U - 1]] < \epsilon'' \tag{4.3.μ}$$

and

$$E = UF'U^{-1}. \tag{4.3.ν}$$

(4.3.μ) implies $[[A_h - U^{-1}A_hU]] = [[A_h - A_hU + A_hU - U^{-1}A_hU]] \leqq |||A_h||| [[1 - U]] + [[1 - U^{-1}]] |||A_hU||| \leqq 2 |||A_h||| \cdot \epsilon''$. Hence by (4.3.$\iota$) $[[B_h' - U^{-1}A_hU]] < (2|||A_h||| + 1)\epsilon''$ and so

$$[[UB_h'U^{-1} - A_h]] < (2|||A_h||| + 1)\epsilon''. \tag{4.3.o}$$

Now we choose $\epsilon'' = \epsilon''(\epsilon) > 0$ as

$$\epsilon'' \doteq \epsilon/(2 \operatorname{Max}_{h=1,\cdots,m} |||A_h||| + 1). \tag{4.3.ξ}$$

Put $\mathbf{N} = \mathbf{N}_3 = U\mathbf{N}^+U^{-1}$. Then $\mathbf{N}$ is a factor in the case (I_q) along with $\mathbf{N}^+$. Thus (i) holds, (ii) holds since $\mathbf{N}^+ \subseteq \mathbf{M}$. $U \in \mathbf{M}$ implies $\mathbf{N} \subseteq \mathbf{M}$. $B_h' \in \mathbf{N}^+$ gives $B_h = UB_hU^{-1} \in \mathbf{N}$, while (4.3.$o$) and (4.3.$\xi$) imply $[[B_h - A_h]] < \epsilon$, so (iii) holds. And finally $F' \in \mathbf{N}^+$ and (4.3.ν) imply $E = UF'U^{-1} \in \mathbf{N}$ so that (iv′) holds.

LEMMA 4.3.3. *If we add to the assumptions of Lemma* 4.3.2 *that*

$$EA_h = A_hE = A_h \tag{α}$$

then we can add to its assertions that

$$EB_h = B_hE = B_h.^{36} \tag{β}$$

PROOF. Apply Lemma 4.3.2, but write B_h' for its B_h. Now put $B_h = EB_h'E$. Then (β) is satisfied and B_h', $E \in \mathbf{N}$ give $B_h \in \mathbf{N}$. By (α), $EA_hE = A_h$. By (iii) of Lemma 4.3.2, $[[B_h' - A_h]] < \epsilon$. Hence $[[E(B_h' - A_h)E]] < \epsilon$ or $[[B_h - A_h]] < \epsilon$. Thus (iii) holds for B_h as well as for B_h'.

The other assertions ((i), (ii), (iv′)) are not affected by our replacing B_h' by B_h. Thus the proof is completed.

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[36] With the same n_3, N_3 as in that lemma.

LEMMA 4.3.4. *If we add to the assumption of Lemma* 4.3.3 *that* $E \in \mathbf{N}^0$ *where* $\mathbf{N}^0$ *is a factor in case* (I_p), *then we can add to its assertions*

(v) $\mathbf{N}^0 \subseteq \mathbf{N}$.[33, 36]

PROOF. Apply Lemma 4.3.3 but write $\mathbf{N}^+$ for its $\mathbf{N}$.

The considerations at the beginning of the proof of (i) in Lemma 4.1.2 show again that $D_{\mathbf{M}}(G) = D_{\mathbf{N}^0}(G)$ for all projections $G \in \mathbf{N}^0 (\subseteq \mathbf{M})$.

Hence in particular $D_{\mathbf{N}_0}(E) = D_{\mathbf{M}}(E) = 1/p$. Then the argument used in proof of Lemma 2.6.2 establishes the existence of a system of p-order matrix units $W^0_{u,v}$, $u, v = 1, \cdots, p$ in $\mathbf{N}^0$ with $\sum_{u=1}^{p} W^0_{u,u} = 1$ and $W^0_{1,1} = E$. Thus in terms of (β) of Lemma 4.1.1,

$(4.3.\pi)$ There exists a matrix base $W^0_{u,v}$, $u, v = 1, \cdots, p$ of $\mathbf{N}^0$ with $W^0_{1,1} = E$.

On the other hand, the argument of Lemma 2.6.3 applied to $\mathbf{N}^+$ establishes the existence of a system of p-order matrix units $W^+_{u,v}$ $u, v = 1, \cdots, p$ in $\mathbf{N}^+$ with $\sum_{u=1}^{p} W^+_{u,u} = 1$ and $W^+_{1,1} = E$. (Observe that this is not a matrix base of $\mathbf{N}^+$ since $\mathbf{N}^+$ is in case (I_q) and not (I_p)!).

Now put $U = \sum_{u=1}^{p} W^0_{u,1} W^+_{1,u}$. As $W^0_{u,1} \in \mathbf{N}^0 \subseteq \mathbf{M}$, $W^+_{1,u} \in \mathbf{N} \subseteq \mathbf{M}$ so $U \in \mathbf{M}$. One verifies by direct computation that $U^*U = UU^* = 1$, i.e. U is unitary. Also

$$(4.3.\rho) \qquad UW^+_{u,v} = W^0_{u,v}U \quad \text{or} \quad W^0_{u,v} = UW^+_{u,v}U^{-1}$$

and, since $W^0_{1,1} = W^+_{1,1} = E$

$$(4.3.\sigma) \qquad EU = UE = E.$$

Apply the mapping

$$(4.3.\tau) \qquad X \to UXU^{-1}$$

to $\mathbf{N}^+$. This carries $\mathbf{N}^+$ into $\mathbf{N} = U\mathbf{N}^+U^{-1}$. Now we see: $\mathbf{N}$ is a factor in case (I_q) along with $\mathbf{N}^+$; thus (i) holds. $\mathbf{N}^+ \subseteq \mathbf{M}$, $U \in \mathbf{M}$ imply $\mathbf{N} \subseteq \mathbf{M}$; so (ii) holds. By (β) in Lemma 4.3.3 and $(4.3.\sigma)$, the mapping $(4.3.\tau)$ leaves all B_h fixed; hence they belong to $\mathbf{N}$ as well as to $\mathbf{N}^+$. Now (iii) and (β) in Lemma 4.3.2 are unaffected and hence hold in the new case. By $(4.3.\rho)$ the mapping $(4.3.\tau)$ carries $W^+_{u,v}$ into $W^0_{u,v}$. Hence $W^+_{u,v} \in \mathbf{N}^+$ gives $W^0_{u,v} \in \mathbf{N}$. Hence by $(4.3.\pi)$ and (ii) in Lemma 4.1.1, every $A \in \mathbf{N}^0$ has the form $A = \sum_{u=1}^{p} \sum_{v=1}^{p} a_{v,u} W^0_{u,v}$ so $A \in \mathbf{N}$. Thus $\mathbf{N}^0 \subseteq \mathbf{N}$ which is the desired relation (v).

Thus the proof is completed.

LEMMA 4.3.5. *Assume that* $\mathbf{M}$ *is approximately finite* (A). *Given any* $A_1, \cdots, A_m \in \mathbf{M}$ *any* $p = 1, 2, \cdots$ *and any* $\epsilon > 0$, *then there exists an* $n_4 = n_4(A_1, \cdots, A_m, p, \epsilon)$ *such that for every* $q \geqq n_4$ *which is divisible by* p *and every*

factor $\mathbf{N}^0 \subseteq \mathbf{M}$ *in case* (I_p), *there exists a factor* $\mathbf{N} = \mathbf{N}_4(q, A_1, \cdots, A_m, \mathbf{N}^0, p, \epsilon)$ *with the following properties*:

(i) $\mathbf{N}$ *is in case* (I_q).

(ii) $\mathbf{N} \subseteq \mathbf{M}$.

(iii) *There exist* $B_1, \cdots, B_m \in \mathbf{N}$ *with* $[[B_h - A_h]] < \epsilon$ *for* $h = 1, \cdots, m$.

(v) $\mathbf{N}^0 \subseteq \mathbf{N}$.

(*Note.* This lemma is quite similar to Lemma 4.3.4. However, (α) has been dropped from the assumptions and (β) from the conclusion. Thus E is no longer mentioned.)

PROOF. Choose a matrix base $W_{u,v}$, $u, v = 1, \cdots, p$ of $\mathbf{N}^0$ and let $E = W_{1,1}$. Put $A_{u,v;h} = W_{1,u}A_hW_{v,1}$.

We have $D_{\mathbf{M}}(E) = D_{\mathbf{M}}(W_{1,1}) = 1/p$ (cf. the argument of (2.7.α)). Clearly $EA_{u,v;h} = A_{u,v;h}E = A_{u,v;h}$.

Now apply Lemma 4.3.4 with the mp^2 operators $A_{u,v;h}$ $h = 1, \cdots, m$, $u, v = 1, \cdots, p$ in place of $A_1, \cdots, A_m$ and ϵ/p^2 in place of ϵ.

Denote the m_3, $\mathbf{N}_3$ which obtain in this way by n_4, $\mathbf{N}_4$. Write $\mathbf{N} = \mathbf{N}_4$. Denote the operators which correspond to the $A_{u,v;h}$ (as the B_h do to the A_h in Lemma 4.3.4) by $B_{u,v;h}$.

Clearly:

$$A_h = \sum_{u=1}^{p} \sum_{v=1}^{p} W_{u,1} A_{u,v;h} W_{1,v}. \tag{4.3.υ}$$

Let

$$B_h = \sum_{u=1}^{p} \sum_{v=1}^{p} W_{u,1} B_{u,v;h} W_{1,v}. \tag{4.3.φ}$$

Now we argue as follows:

(i), (ii) and (v) in Lemma 4.3.4 are unaffected. $B_{u,v;h} \in \mathbf{N}$ and $W_{u,v} \in \mathbf{N}^0 \subseteq \mathbf{N}$ imply $B_h \in \mathbf{N}$. We have $[[B_{u,v;h} - A_{u,v;h}]] < \epsilon/p^2$. Hence $[[W_{u,1}(B_{u,v;h} - A_{u,v;h})W_{1,v}]] < \epsilon/p^2$ and $[[\sum_{u=1}^{p} \sum_{v=1}^{p} W_{u,1}(B_{u,v;h} - A_{u,v;h})W_{1,v}]] < \epsilon$. Thus Thus (4.3.$\upsilon$) and (4.3.$\varphi$) and this inequality yields $[[B_h - A_h]] < \epsilon$. So (iii) holds and the proof is complete.

§4.4 We now use the results of the preceding two sections to compare the two notions of approximate finiteness which we have introduced.

LEMMA 4.4.1. *Assume that* $\mathbf{M}$ *is approximately finite* (A) *and that the sequence* $[p_1, p_2, \cdots]$ *fulfills* (i), (ii) *in Lemma* 4.1.3. *Then* $[p_1, p_2, \cdots]$ *possesses a subsequence* $[p_{1'}, p_{2'}, \cdots]$ *such that* $\mathbf{M}$ *is approximately finite of type* $[p_{1'}, p_{2'}, \cdots]$.

PROOF. By [2], pp. 387–388, there exists a sequence $A_1, A_2, \cdots \in \mathbf{M}$ which is everywhere dense in $\mathbf{M}$ in the sense of strong convergence. Put $p_0 = 1$ and $\mathbf{N}_0 = (\alpha 1)$. This is a factor $\subseteq \mathbf{M}$ in case (I_1). We shall choose by induction for every $n = 1, 2, \cdots$, a $p_{n'}$ and a factor $\mathbf{N}_n \subseteq \mathbf{M}$ in the case $(I_{p_{n'}})$. If for any $n = 1, 2, \cdots$, $p_{(n-1)'}$ and $\mathbf{N}_{n-1}$ have already been chosen, then we choose $p_{n'}$ and $\mathbf{N}_n$ in the following way. Apply Lemma 4.3.5 to $A_1, \cdots, A_n$, $p_{(n-1)'}$, $1/n$, $\mathbf{N}_{n-1}$ in place of its $A_1, \cdots, A_m$, p, ϵ, $\mathbf{N}^0$. Choose $p_{n'}$ from the sequence $[p_1, p_2, \cdots]$ with $p_{n'} > \mathrm{Max}\ (n_4, p_{(n-1)'})$. Thus $p_{n'}$ is divisible by $p_{(n-1)'}$.

Put $q = p_{n'}$ and then put $\mathbf{N}_n = \mathbf{N}$ (from the above application of Lemma 4.3.5).

Thus $[p_{1'}, p_{2'}, \cdots]$ is a subsequence of $[p_1, p_2, \cdots]$. We have $\mathbf{N}_{n-1} \subseteq \mathbf{N}_n$, hence

$$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots . \tag{4.4.α}$$

$\mathbf{N}_n$ is a factor $\subseteq \mathbf{M}$ of class $(I_{p_{n'}})$. Clearly

$$\mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) \subseteq \mathbf{M}. \tag{4.4.β}$$

Now for $n \geqq h$, there exists a $B_h^{(n)} \epsilon \mathbf{N}_n$ with $[[B_h^{(n)} - A_h]] < 1/n$. So A_h is a metric limit of $\mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots)$ and hence, by Theorem I, $A_h \epsilon \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots)$. Thus $A_1, A_2, \cdots \epsilon \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots)$. Since the $A_1, A_2, \cdots$ are everywhere dense in $\mathbf{M}$ in the sense of the strong convergence, this and (4.4.β) yield

$$\mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) = \mathbf{M}. \tag{4.4.γ}$$

(4.4.α) and (4.4.γ) and our observations establish that $\mathbf{M}$ is approximately finite of type $[p_{1'}, p_{2'}, \cdots]$ as stated.

We need one more auxiliary lemma. In view of a later application, we shall prove it in a stronger form than that immediately needed.

LEMMA 4.4.2. *Let* $\mathbf{N}$, $\mathbf{O}$ *be two factors in the case* (I_p) *and* (I_q) *or* (II_1) *respectively. Let* p *be a divisor of* r *and (if* q *exists)* r *a divisor of* q. *Suppose* $\mathbf{N} \subseteq \mathbf{O}$. *Then there exists a factor* $\mathbf{R}$ *in case* (I_r) *such that* $\mathbf{N} \subseteq \mathbf{R} \subseteq \mathbf{O}$.

PROOF. By Lemma 2.6.2 or Lemma 2.6.3, since $\mathbf{O}$ is in case (II_1) or in the case (I_q), q divisible by r, there exists a factor $\mathbf{R} \subseteq \mathbf{O}$ in the case (I_r) with a matrix base $W_{o,w}^+$, $o, w = 1, \cdots, r$. Choose also a matrix base $W_{u,v}^0$, $u,v = 1, \cdots, p$ of N^+. Thus $W_{o,w}^+$ is a system of r-order matrix units with $\sum_{o=1}^{r} W_{o,o}^+ = 1$ and $W_{u,v}^0$ is a system of p-order matrix units with $\sum_{u=1}^{p} W_{u,u}^0 = 1$.

Put

$$W_{u,v}^{++} = \sum_{i=0}^{r/p} W_{(u-1)r/p+i,(v-1)r/p+i}^+ . \tag{4.4.δ}$$

Then one verifies by direct calculation that the $W_{u,v}^{++}$, $u, v = 1, \cdots, p$ form a system of p-order units with $\sum_{u=1}^{p} W_{u,u}^{++} = 1$. One can readily prove that $D_{\mathbf{O}}(W_{1,1}^{++}) = 1/p = D_{\mathbf{O}}(W_{1,1}^0)$ in the standard normalization. Thus there exists a partially isometric $V \epsilon \mathbf{O}$ such that $V^*V = W_{1,1}^{++}$, $VV^* = W_{1,1}^0$. Now put

$$U = \sum_{u=1}^{p} W_{u,1}^0 V W_{1,u}^{++} .$$

Clearly $U \epsilon \mathbf{O}$. One verifies by direct computation that $U^*U = UU^* = 1$ and hence that U is unitary. Also that

$$UW_{u,v}^{++} = W_{u,v}^0 U \quad \text{or} \quad W_{u,v}^0 = UW_{u,v}^{++}U^{-1}. \tag{4.4.ϵ}$$

Hence the replacement of $\mathbf{R}$ by $U\mathbf{R}U^{-1}$ and of the system $W_{o,w}^+$ by the system $UW_{o,w}^+U^{-1}$ leaves all their properties unaffected, but it carries $W_{u,v}^{++}$ into $W_{u,v}^0 = UW_{u,v}^{++}U^{-1}$. Consequently, we may assume that we had $W_{u,v}^{++} = W_{u,v}^0$ i.e.

$$W_{u,v}^0 = \sum_{i=1}^{r/p} W_{(u-1)r/p+i,(v-1)r/p+i}^+ . \tag{4.4.ζ}$$

to begin with.

Thus all $W^0_{u,v} \in \mathbf{R}$, hence $\mathbf{N} \subseteq \mathbf{R}$. We know already that $\mathbf{R}$ is a factor in the case (I_r) and that $\mathbf{R} \subseteq \mathbf{O}$.

Thus the proof is completed.

We can now conclude our deductions from Def. 4.3.1.

LEMMA 4.4.3. *Under the same assumptions as Lemma* 4.4.1, $\mathbf{M}$ *is approximately finite of type* $[p_1, p_2, \cdots]$.

PROOF. Apply Lemma 4.4.1. Denote its $\mathbf{N}_1, \mathbf{N}_2, \cdots$ by $\mathbf{N}_{1'}, \mathbf{N}_{2'}, \cdots$. Write $0' = 0$, $\mathbf{N}_{0'} = \mathbf{N}_0 = (\alpha \cdot 1)$.

Thus $\mathbf{N}_n$ is now defined for $n = 0', 1', 2', \cdots$ only. Consider a fixed $m' = 1', 2', \cdots$. We have $\mathbf{N}_{(m-1)'} \subseteq \mathbf{N}_{m'}$ and that $\mathbf{N}_{(m-1)'}, \mathbf{N}_{m'}$ are factors in the cases $(\mathrm{I}_{p_{(m-1)'}})$, $(\mathrm{I}_{p_{m'}})$ respectively. Each $p_{(m-1)'}, p_{(m-1)'+1}, \cdots, p_{m'-1}, p_{m'}$ is a divisor of its successor. Hence we obtain by repeated applications of Lemma 4.4.2 a succession of factors $\mathbf{N}_{(m-1)'+1}, \mathbf{N}_{(m-1)'+2}, \cdots, \mathbf{N}_{m'-1}$, with these properties:

$$\mathbf{N}_{(m-1)'} \subseteq \mathbf{N}_{(m-1)'+1} \subseteq \cdots \subseteq \mathbf{N}_{m'-1} \subseteq \mathbf{N}_{m'}$$

and $\mathbf{N}_n$ $(n = (m-1)' + 1, \cdots, m' - 1)$ is in the case (I_{p_n}).

So we have defined $\mathbf{N}_n$ for all $n = 1, 2, \cdots$. We have

$$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots, \tag{4.4.η}$$

and $\mathbf{N}_n$ is a factor in the case (I_{p_n}). Finally, by (4.4.η) and Lemma 4.4.1, we have:

$$\mathbf{R}(\mathbf{N}_n ; n = 1, 2, \cdots) = \mathbf{R}(\mathbf{N}_{m'} ; m = 1, 2, \cdots) = \mathbf{M} \tag{4.4.θ}$$

(4.4.η), (4.4.θ) and our other observations establish that $\mathbf{M}$ is approximately finite of type $[p_1, p_2, \cdots]$ as stated.

§4.5 We proceed now to the second operation announced at the beginning of §4.3.

DEFINITION 4.5.1. *A ring* $\mathbf{N}$ *is of finite order if it possesses a finite linear basis, i.e. a (fixed finite) system,* $A^0_1, \cdots, A^0_q$ *such that* $\mathbf{N}$ *is the set of all*

$$A = \sum_{u=1}^{q} x_u A^0_u \qquad (x_1, \cdots, x_q \text{ any complex numbers}).$$

DEFINITION 4.5.2. $\mathbf{M}$ *is approximately finite (B) if this is true:*

Given any $A_1, \cdots, A_m \in \mathbf{M}$ *and any* $\epsilon > 0$ *there exists a ring* $\mathbf{N}$ *with the following properties*

(i) $\mathbf{N}$ *is of finite order*

(ii) $\mathbf{N} \subseteq \mathbf{M}$

(iii) *There exists* $B_1, \cdots, B_m \in \mathbf{N}$ *with*

$$[[B_h - A_h]] < \epsilon \quad \text{for} \quad h = 1, \cdots, m.$$

LEMMA 4.5.1. *If* $\mathbf{M}$ *is approximately finite of type* $[p_1, p_2, \cdots]$ *then it is also approximately finite (B).*

PROOF. Suppose $A_1, \cdots, A_m \in \mathbf{M}$ and a $\epsilon > 0$ are given. Form the

$\mathbf{N}_1, \mathbf{N}_2, \cdots$ of Def. 4.1.1. By (ii) in Lemma 4.1.4, there exists for each $h = 1, \cdots, m$ an $\mathbf{N}_{n_h}$ such that for some $B_h \in \mathbf{N}_{n_h}$

$$[[B_h - A_h]] < \epsilon. \tag{4.5.α}$$

Put $n = \text{Max}\ (n_1, \cdots, n_m)$ then $B_h \in \mathbf{N}_{n_h} \subseteq \mathbf{N}_n$ i.e.

$$B_1, \cdots, B_m \in \mathbf{N}_n . \tag{4.5.β}$$

Now $\mathbf{N}_n$ is a factor in the case (I_{p_n}). By (ii) in Lemma 4.1.1 it is a ring of finite order. This, together with (4.5.α) and (4.5.β), completes the proof.

Comparison of Lemmas 4.4.3 and 4.5.1 shows that if we can derive approximate finiteness (A) from (B), then the equivalence of all kinds of approximate finiteness is established. This we accomplish in the next section. In the present section we obtain certain preliminary lemmas on rings of finite order; indeed we shall determine the structure of these.[37] Thus for the present section $\mathbf{N}$ is to be a ring of finite order.

LEMMA 4.5.2. (i) *There is a maximum r such that there is a system $E_1, \cdots, E_r$ of mutually orthogonal projections $\neq 0$ and $\in \mathbf{N}$.*

(ii) *For such a system (with r maximum) $E = \sum_{s=1}^{r} E_s$ is the unit of $\mathbf{N}$.*

(iii) *Every E_s in such a system (with r maximum) is minimal (with respect to $\mathbf{N}$) in the sense of* [5], *p.* 143, *Def.* 5.1.2.

PROOF. Ad (i): Any such system is necessarily linearly independent. Hence the number of its elements must be $\leqq$ the q of Def. 4.5.1. Hence the numbers for which there is such a system have a maximum.

Ad (ii): Let E be the unit of $\mathbf{N}$. $\sum_{s=1}^{r} E_s$ is a projection in $\mathbf{N}$; hence $\sum_{s=1}^{r} E_s \leqq E$. So $E_{r+1} = E - \sum_{s=1}^{r} E_s$ is a projection in $\mathbf{N}$ which is orthogonal to the original E_s. Hence the assumed maximum property of r necessitates $E_{r+1} = 0$, i.e. $E = \sum_{s=1}^{r} E_s$.

Ad (iii): Consider a projection $F \leqq E_s$ in $\mathbf{N}$. Then $E_s' = F$ and $E_s'' = E_s - F$ are mutually orthogonal projections in $\mathbf{N}$ which are also orthogonal to all $E_{s'}$, $s' \neq s$. Hence the assumed maximum property of r necessitates $E_s' = 0$ or $E_s'' = 0$ i.e. $F = 0$ or E_s. This however is the defining property of minimality.

REMARK. The center $\mathbf{N} \cdot \mathbf{N}'$ of $\mathbf{N}$ is of finite order along with $\mathbf{N}$. Hence we can apply the above lemma to it, thus obtaining a system $\bar{E}_1, \cdots, \bar{E}_{\bar{r}}$.

By this lemma, $\bar{E} = \sum_{s=1}^{\bar{r}} \bar{E}_s$ is the unit of $\mathbf{N} \cdot \mathbf{N}'$, hence by [2], p. 393, Theorems 3 and 4, it is also the unit of $\mathbf{N}$.

LEMMA 4.5.3. (i) *$\bar{E}_s$ (cf. above) is such that its range $\overline{\mathfrak{M}}_s$ reduces $\mathbf{N}$, $\mathbf{N}'$, $\mathbf{N} \cdot \mathbf{N}'$*

$$(\mathbf{N} \cdot \mathbf{N}')_{(\bar{\mathfrak{M}}_s)} = (\bar{E}_s)_{(\bar{\mathfrak{M}}_s)} \tag{ii}$$

PROOF. Ad (i): This is implied by $\bar{E}_s \in \mathbf{N} \cdot \mathbf{N}'$.

Ad (ii): This is equivalent to the statement $\bar{E}_s A = A\bar{E}_s = \alpha \bar{E}_s$ for all $A \in \mathbf{N} \cdot \mathbf{N}'$. Put $\bar{E}_s A = A\bar{E}_s = A'$. Clearly $A' \in \mathbf{N} \cdot \mathbf{N}'$, $\bar{E}_s A' = A'\bar{E}_s = A'$. Since $\bar{E}_s$ is a minimal projection in $\mathbf{N} \cdot \mathbf{N}'$ this implies $A' = \alpha \bar{E}_s$ (cf. the last part of the proof of Lemma 5.1.3 in [5], p. 144).

LEMMA 4.5.4. $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ *is a factor (in* $\bar{\mathfrak{M}}_s$*) in a case* (I_{q_s}) $(q_s = 1, 2, \cdots)$.

PROOF. If A' is in the center of $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$, $A'\bar{E}_s$ defines an operator A in $\mathfrak{H}$ which is readily seen to be in the center of $\mathbf{N}$ and for which $A\bar{E}_s = \bar{E}_s A = A$. Thus the center of $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ is included in $(\mathbf{N}\cdot\mathbf{N}')_{(\bar{\mathfrak{M}}_s)}$. The converse inclusion is obvious.

It follows that the center of $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ is $(\mathbf{N}\cdot\mathbf{N}')_{(\bar{\mathfrak{M}}_s)}$, and hence $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ is a factor by (ii) in Lemma 4.5.3.

$\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ is of finite order along with $\mathbf{N}$. Therefore Lemma 4.5.2 implies that $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ contains a minimal projection. Thus $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ is a direct factor (cf. [5], pp. 150, 173, Theorem IV and Lemma 8.6.1). Obviously, since it is of finite order, $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ is not in a case (I_∞), and hence it is in a case (I_{q_s}), $q_s = 1, 2, \cdots$. This completes the proof.

LEMMA 4.5.5. $\mathbf{N}$ *possesses a finite linear base which consists of the operators* $W^s_{u,v}$, $s = 1, \cdots, \bar{r}$, $u, v = 1, \cdots, q_s$ *where for any fixed* s *the* $W^s_{u,v}$, $u, v = 1, \cdots, q_s$ *are a system of* q_s*-order matrix units with* $\bar{E}_s = \sum_{u=1}^{q_s} W^s_{u,u}$. *Thus for the* q *of Def.* 4.5.1, $q = \sum_{s=1}^{\bar{r}} (q_s)^2$.

PROOF. In virtue of Lemma 4.5.4, we may apply Lemma 4.1.1 and Remark 2 after it to $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$. So $\mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ possesses a finite linear base of operators $\tilde{W}^s_{u,v}$ (in $\bar{\mathfrak{M}}_s$), $u, v = 1, \cdots, q_s$ which form a system of q_s-order matrix units in $\bar{\mathfrak{M}}_s$ with $\bar{E}_{s(\bar{\mathfrak{M}}_s)} = \sum_{u=1}^{q_s} \tilde{W}^s_{u,u}$.

Since $\tilde{W}^s_{u,v} \in \mathbf{N}_{(\bar{\mathfrak{M}}_s)}$ there is a $W^{s0}_{u,v}$ in $\mathbf{N}$ for which the part in $\bar{\mathfrak{M}}_s$ is $\tilde{W}^s_{u,v}$. Let $W^s_{u,v} = W^{s0}_{u,v}\bar{E}_s$. Since $\bar{E}_s \in \mathbf{N}\cdot\mathbf{N}'$, we have $W^s_{u,v} \in \mathbf{N}$ and also $W^s_{u,v} = W^s_{u,v}\bar{E}_s = \bar{E}_s W^s_{u,v}$. Clearly $(W^s_{u,v})_{(\bar{\mathfrak{M}}_s)} = \tilde{W}^s_{u,v}$ and one can readily show that the $\tilde{W}^s_{u,v}$ form a system of q_s-order matrix units in $\mathfrak{H}$ with $\bar{E}_s = \sum_{u=1}^{q_s} W^s_{u,u}$.

Consider an $A_s \in \mathbf{N}$ with $\bar{E}_s A_s = A_s \bar{E}_s = A_s$. Then A_s has the part $A_{s(\bar{\mathfrak{M}}_s)}$ in $\bar{\mathfrak{M}}_s$ and 0 in $\bar{\mathfrak{M}}_s^{\perp}$. In the sense then that $\bar{E}_s$ is a transformation from $\mathfrak{H}$ to $\bar{\mathfrak{M}}_s$, $A_s = A_{s(\bar{\mathfrak{M}}_s)}\bar{E}_s$ and also $W^s_{u,v} = \tilde{W}^s_{u,v}\bar{E}_s$. As $A_{s(\bar{\mathfrak{M}}_s)} \in \mathbf{N}_{(\bar{\mathfrak{M}}_s)}$, we have, from the first paragraph of the proof, that $A_{s(\bar{\mathfrak{M}}_s)} = \sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v}\tilde{W}^s_{u,v}$ and hence $A_s = A_{s(\bar{\mathfrak{M}}_s)}\bar{E}_s = \sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v}\tilde{W}^s_{u,v}\bar{E}_s = \sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v}W^s_{u,v}$.

Consider now any $A \in \mathbf{N}$. Put $A_s = \bar{E}_s A$ which also equals $A\bar{E}_s$ since $\bar{E}_s \in \mathbf{N}\cdot\mathbf{N}'$. By the above, $A_s = \sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v}W^s_{u,v}$. By the remark after Lemma 4.5.2, we have

$$A = \bar{E}A = \sum_{s=1}^{\bar{r}} \bar{E}_s A = \sum_{s=1}^{\bar{r}}\sum_{u=1}^{q_s}\sum_{v=1}^{q_s} a^s_{u,v} W^s_{u,v}.$$

Thus the $W^s_{u,v}$, $s = 1, \cdots, \bar{r}$, $u, v = 1, \cdots, q_s$ form indeed a finite linear basis of $\mathbf{N}$.

We now prove a lemma which possesses a certain interest of its own. It applies to any ring of finite order $\mathbf{N} \subseteq \mathbf{M}$. We assume that a representation of $\mathbf{N}$ in the sense of Lemma 4.5.5 is given, with the $W^s_{u,v}$ as described there.

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[37] The four lemmas which follow are merely an *ad hoc* derivation of Wedderburn's structure theorems for semi-simple algebras—specialized to the present situation. They can also be interpreted as a summary of certain results of unitary representation theory including the theorem of Burnside-Frobenius-Schur for that case.

LEMMA 4.5.6. *Given a (fixed)* $q = 1, 2, \cdots$ *there exists a factor* $\mathbf{O}$ *in case* (I_q) *with* $\mathbf{N} \subseteq \mathbf{O} \subseteq \mathbf{M}$ *if and only if this is true for* $\mathbf{N}$:

$$(4.5.\gamma) \qquad \text{All } qD_{\mathbf{M}}(W^s_{1,1}) \text{ are integers.}$$

PROOF. Assume that an $\mathbf{O}$ with the specified properties exists.

The considerations at the beginning of the proof of (i) in Lemma 4.1.2 show again that $D_{\mathbf{M}}(G) = D_{\mathbf{O}}(G)$ for all projections $G \in \mathbf{O}(\subseteq \mathbf{M})$. Hence in particular, $D_{\mathbf{M}}(W^s_{1,1}) = D_{\mathbf{O}}(W^s_{1,1})$, since $W^s_{1,1}$ is a projection $\in \mathbf{N} \subseteq \mathbf{O}$. But as $\mathbf{O}$ is in case (I_q), $D_{\mathbf{M}}(W^s_{1,1}) = D_{\mathbf{O}}(W^s_{1,1})$ must have a value $(0, 1/q, \cdots, 1)$ i.e. $q \cdot D_{\mathbf{M}}(W^s_{1,1})$ must be an integer. Thus $(4.5.\gamma)$ holds.

Sufficiency: Assume that $\mathbf{N}$ fulfills $(4.5.\gamma)$. Considering Lemma 4.5.5, we must prove this:

$$(4.5.\delta) \quad \begin{cases} \text{Let a system of mutually orthogonal projections } \neq 0 \text{ from } \mathbf{M} \text{ be given:} \\ \bar{E}_1, \cdots, \bar{E}_{\bar{r}}. \text{ For each } s, \text{ let a system of } q_s\text{-order matrix units } W^s_{u,v} \\ \text{from } \mathbf{M} \text{ be given, with } \bar{E}_s = \sum_{u=1}^{q_s} W^s_{u,u} \text{ and } q_s = 1, 2, \cdots. \text{ Let for} \\ \text{a given } q \text{ every } qD_{\mathbf{M}}(W^s_{1,1}) \text{ be an integer. Then there exists a factor} \\ \mathbf{O} \subseteq \mathbf{M} \text{ in a case } (I_q) \text{ such that all } W^s_{u,v} \in \mathbf{O}. \end{cases}$$

We therefore disregard $\mathbf{N}$ completely and deal with $(4.5.\delta)$ alone.

PROOF OF $(4.5.\delta)$. We have $D_{\mathbf{M}}(W^s_{1,1}) = p_s/q$, $p_s = 1, 2, \cdots$. The projections $W^s_{1,1}$ and $W^s_{u,u}$ are equivalent in virtue of the partially isometric $W^s_{u,1}$ (cf. Lemma 2.6.1). Hence

$$(4.5.\epsilon) \qquad D_{\mathbf{M}}(W^s_{u,u}) = p_s/q, \qquad D_{\mathbf{M}}(\bar{E}_s) = p_s q_s/q.$$

We may assume that

$$(4.5.\zeta) \qquad \sum_{s=1}^{\bar{r}} \bar{E}_s = 1.$$

Otherwise we could put $\bar{E}_{\bar{r}+1} = 1 - \sum_{s=1}^{\bar{r}} \bar{E}_s$. Then $\bar{E}_{\bar{r}+1}$ is a projection $\neq 0$. Also $qD_{\mathbf{M}}(\bar{E}_{\bar{r}+1}) = q - \sum_{s=1}^{\bar{r}} qD_{\mathbf{M}}(\bar{E}_s)$ is an integer. Thus we could add $\bar{E}_{\bar{r}+1}$ to the system $\bar{E}_1, \cdots, \bar{E}_{\bar{r}}$ with $q_{\bar{r}+1} = 1$ and $W^{\bar{r}+1}_{1,1} = \bar{E}_{\bar{r}+1}$.

By $(4.5.\epsilon)$ and $(4.5.\zeta)$, $\sum_{s=1}^{\bar{r}} p_s q_s = q$. We put $l_s = \sum_{t=1}^{s} p_t q_t$ for $s = 0, 1, \cdots, \bar{r}$. Then

$$(4.5.\eta) \quad \begin{cases} l_s = l_{s-1} + p_s q_s & \text{for } s = 1, \cdots, \bar{r} \\ l_0 = 0, \qquad l_{\bar{r}} = q. \end{cases}$$

As $\mathbf{M}$ is in case (II_1), by Lemma 2.6.2 there exists a factor $\mathbf{O}^+ \subseteq \mathbf{M}$ in the case (I_q) with its matrix base $W^+_{o,w}$, $o, w = 1, \cdots, q_s$. Thus $W^+_{o,w}$ is a system of q-order matrix units, with $\sum_{u=1}^{q} W^+_{u,u} = 1$.

Put

$$(4.5.\theta) \qquad E^+_s = \sum_{u=l_{s-1}+1}^{l_s} W^+_{u,u} \qquad \text{for } s = 1, \cdots, \bar{r}$$

$$(4.5.\iota) \quad \begin{cases} W^{+s}_{u,v} = \sum_{u=1}^{p_s} W^+_{l_{s-1}+(u-1)p_s+i,\, l_{s-1}+(v-1)p_s+i} \\ \qquad\qquad \text{for } s = 1, \cdots, \bar{r},\ u, v = 1, \cdots, q_s. \end{cases}$$

Then one verifies by a direct computation that the E_s^+ are mutually orthogonal projections $\neq 0$ with $\sum_{s=1}^{\bar{r}} E_s^+ = 1$ and (using Def. 2.6.1) that the $W_{u,v}^{+s}$ form (for any fixed s) a system of q_s-order matrix units with $\sum_{u=1}^{q_s} W_{u,u}^{+s} = E_s^+$. All these operators are in $\mathbf{M}$.

The $W_{u,u}^+$ are equivalent (due to the $W_{u,v}^+$) and since $\sum_{u=1}^{q} W_{u,u}^+ = 1$, $D_{\mathbf{M}}(W_{u,u}^+) = 1/q$. Hence by (4.5.$\iota$), $D_{\mathbf{M}}(W_{1,1}^{+s}) = p_s/q$ and by (4.5.ϵ), $D_{\mathbf{M}}(W_{1,1}^{+s}) = D_{\mathbf{M}}(W_{1,1}^s)$. Consequently there exists a partially isometric operator $V_s \in \mathbf{M}$ with the initial projection $W_{1,1}^{+s}$ and final projection $W_{1,1}^s$.

Now put

$$U = \sum_{s=1}^{\bar{r}} \sum_{u=1}^{q_s} W_{u,1}^s V_s W_{1,u}^{+s}.$$

Clearly $U \in \mathbf{M}$. One verifies, by direct computation, $U^*U = UU^* = 1$ which implies that U is unitary, and similarly that

$$UW_{u,v}^{+s} = W_{u,v}^s U \quad \text{or} \quad W_{u,v}^s = UW_{u,v}^+ U^{-1}. \tag{4.5.κ}$$

$\mathbf{O} = U\mathbf{O}^+U^{-1}$ is a factor $\subseteq \mathbf{M}$ and in case (I_q) along with $\mathbf{O}^+$. By (4.5.ι), $W_{u,v}^{+s} \in \mathbf{O}^+$ and hence by (4.5.κ) $W_{u,v}^s = UW_{u,v}^{+s}U^{-1} \in U\mathbf{O}^+U^{-1} = \mathbf{O}$. Thus all $W_{u,v}^s \in \mathbf{O}$.

§4.6 We are now in a position to complete our proof of the relationships between the various kinds of approximate finiteness.

LEMMA 4.6.1. *Suppose* $\mathbf{N}$, $\mathbf{M}$, $\bar{E}_1, \cdots, \bar{E}_r$ *and the* $W_{u,v}^s$ *are as in Lemma* 4.5.5 *with* $\bar{E} = \sum_{s=1}^{\bar{r}} \bar{E} = 1$. *For any* $\epsilon > 0$ *there exists an* $n_5 = n_5(\mathbf{N}, \epsilon)$ *such that for every* $q \geqq n_5$ *there exists a projection* $E = E(q, \mathbf{N}, \epsilon)$ *which is* $\in \mathbf{M} \cdot \mathbf{N}'$ *and has the following properties*:

(i) $D_{\mathbf{M}}(1 - E) < \epsilon$

(ii) *All* $qD_{\mathbf{M}}(EW_{1,1}^s)$ *are* $\neq 0$ *and integers.*

PROOF. Given a $q = 1, 2, \cdots$ we wish to choose for each $s = 1, \cdots, \bar{r}$ a $k_s = 1, 2, \cdots$ with

$$q(D_{\mathbf{M}}(W_{1,1}^s) - \epsilon/\textstyle\sum_{s=1}^{\bar{r}} q_s) < k_s \leqq qD_{\mathbf{M}}(W_{1,1}^s). \tag{4.6.α}$$

This is certainly feasible if $q \cdot D_{\mathbf{M}}(W_{1,1}^s) \geqq 1$ and $q\epsilon/\sum_{s=1}^{\bar{r}} q_s \geqq 1$ for $s = 1, \cdots, \bar{r}$. We assume therefore that $q \geqq n_5$ with

$$n_5 = n_5(\mathbf{N}, \epsilon) = \mathrm{Max}\,(\textstyle\sum_{s=1}^{\bar{r}} q_s/\epsilon,\ 1/D_{\mathbf{M}}(W_{1,1}^s),\ s = 1, \cdots, \bar{r}). \tag{4.6.β}$$

Now (4.6.α) and $k_s \neq 0$ give

$$D_{\mathbf{M}}(W_{1,1}^s) - \epsilon/\textstyle\sum_{s=1}^{\bar{r}} q_s < k_s/q \leqq D_{\mathbf{M}}(W_{1,1}^s), \qquad k_s/q \neq 0.$$

As $\mathbf{M}$ is in case (II_1), this implies that we can find a projection $G^s \in \mathbf{M}$ with

$$D_{\mathbf{M}}(G^s) = k_s/q \neq 0 \tag{4.6.γ}$$

and

$$G^s \leqq W_{1,1}^s \tag{4.6.δ}$$

and that then

$$D_{\mathbf{M}}(W_{1,1}^s - G_s) < \epsilon/\textstyle\sum_{s=1}^{\bar{r}} q_s. \tag{4.6.ϵ}$$

$W^s_{u,1}$ is a partially isometric operator with initial projection $W^s_{1,1}$ and final projection $W^s_{u,u}$. Its mapping carries therefore the $G^s \leqq W^s_{1,1}$ into a projection $G^s_u \in \mathbf{M}$ with

$$G^s_u \leqq W^s_{u,u}. \tag{4.6.ζ}$$

The same mapping carries (4.6.ϵ) into

$$D_{\mathbf{M}}(W^s_{u,u} - G^s_u) < \epsilon/\textstyle\sum_{s=1}^{\tilde{r}} q_s. \tag{4.6.η}$$

Finally $W^s_{u,v} \cdot W^s_{v,1} = W^s_{u,1}$ implies that the mapping of $W^s_{u,v}$ carries G^s_v into G^s_u. This may be written

$$W^s_{u,v} G^s_v = G^s_u W^s_{u,v}. \tag{4.6.θ}$$

For $s \neq t$ or $v \neq w$, $W^s_{v,v}$ the initial projection of $W^s_{u,v}$ is orthogonal to $W^t_{w,w}$ hence, *a fortiori*, to $G^t_w \leqq W^t_{w,w}$. So we have

$$W^s_{u,v} G^t_w = 0 \qquad \text{for } s \neq t \text{ or } v \neq w. \tag{4.6.ι}$$

$$G^t_w W^s_{u,v} = 0 \qquad \text{for } s \neq t \text{ or } u \neq w \tag{4.6.κ}$$

Now put $E = \sum_{s=1}^{\tilde{r}} \sum_{u=1}^{q_s} G^s_u$. Clearly $E \in \mathbf{M}$ and since $\sum_{s=1}^{\tilde{r}} \sum_{u=1}^{q_s} W^s_{u,u} = \sum_{s=1}^{\tilde{r}} \tilde{E}_s = 1$ we have $1 - E = \sum_{s=1}^{\tilde{r}} \sum_{u=1}^{q_s} (W^s_{u,u} - G^s_u)$. Hence (4.6.$\eta$) yields

$$D_{\mathbf{M}}(1 - E) < \epsilon. \tag{4.6.λ}$$

(4.6.θ), (4.6.ι) and (4.6.κ) give

$$W^s_{u,v} E = E W^s_{u,v} (= W^s_{u,v} G^s_v = G^s_u W^s_{u,v}).$$

So E commutes with $W^s_{u,v}$ and hence by Lemma 4.5.5 with all of $\mathbf{N}$. Since $E \in \mathbf{M}$ this yields

$$E \in \mathbf{M} \cdot \mathbf{N}'. \tag{4.6.μ}$$

Finally, direct computation shows that

$$E W^s_{1,1} = G^s_1 = G^s. \tag{4.6.ν}$$

Our assertions now follow immediately. (4.6.μ) gives the preliminary one, (4.6.λ) implies (i), and (4.6.ν) and (4.6.γ) yield (ii). Thus the lemma is proven.

It is now possible to take the final step.

LEMMA 4.6.2. *If* $\mathbf{M}$ *is approximately finite* (*B*), *then it is also approximately finite* (*A*).

PROOF. We must verify the criteria of Def. 4.3.1 for $\mathbf{M}$.

Let therefore a system $A_1, \cdots, A_m \in \mathbf{M}$ and an $\epsilon > 0$ be given. Apply Def. 4.5.2 to these $A_1, \cdots, A_m$ and to $\epsilon/2$ thus obtaining the ring $\mathbf{N}$ and $B_1, \cdots, B_m \in \mathbf{N}$ described there. Thus

$$[[B_s - A_s]] < \epsilon/2. \tag{4.6.o}$$

$\mathbf{N}$ is a ring of finite order $\subseteq \mathbf{M}$. We form $\bar{E}_1, \cdots, \bar{E}_{\bar{r}}$ and the $W^s_{u,v}$ in the sense of Lemma 4.5.5 for this $\mathbf{N}$. To these and to

$$\epsilon' = (\epsilon/2 \text{ Max } (|||B_s||| ; s = 1, \cdots, \bar{r}))^2 \tag{4.6.π}$$

we apply Lemma 4.6.1, thus obtaining $n_5 = n_5(\mathbf{N}, \epsilon')$. For any $q \geqq n_5$ we form the projection $E = E(q, \mathbf{N}, \epsilon')$ of Lemma 4.6.1.

Accordingly we have $E \in \mathbf{M}' \cdot \mathbf{N}$ and $\mathbf{N}_E = E \cdot \mathbf{N} = \mathbf{N}E$ is a finite order ring $\subseteq \mathbf{M}$ along with $\mathbf{N}$. Thus $E\bar{E}_1, \cdots, E\bar{E}_{\bar{r}}$ and the $EW^s_{u,v}$ perform for $\mathbf{N}_E$ the functions of $\bar{E}_1, \cdots, \bar{E}_{\bar{r}}$ and the $W^s_{u,v}$ for $\mathbf{N}$ (cf. Lemma 4.5.5). By (ii) in Lemma 4.6.1 all $qD_{\mathbf{M}}(EW^s_{1,1})$ are integers. Thus Lemma 4.5.6 is applicable to $\mathbf{N}_E$ and we obtain a factor $\mathbf{O}$ in case (I_q) with

$$\mathbf{N}_E \subseteq \mathbf{O} \subseteq \mathbf{M}. \tag{4.6.ρ}$$

$B_s \in \mathbf{N}$ implies

$$C_s = EB_s \in \mathbf{N}_E \subseteq \mathbf{O}. \tag{4.6.σ}$$

Now (i) in Lemma 4.6.1 gives $D_{\mathbf{M}}(1 - E) < \epsilon'$. Hence $[[1 - E]] < \epsilon'^{\frac{1}{3}}$[5] and therefore

$$[[B_s - C_s]] = [[B_s(1 - E)]] < |||B_s||| \, \epsilon'^{\frac{1}{3}}.$$

Hence by (4.6.π)

$$[[B_s - C_s]] < \epsilon/2. \tag{4.6.τ}$$

Combining (4.6.σ) and (4.6.τ) gives

$$[[C_s - A_s]] < \epsilon. \tag{4.6.υ}$$

Thus we can satisfy Def. 4.3.1 by putting $n_1 = n_1(A_1, \cdots, A_m ; \epsilon) = n_5(\mathbf{N}, \epsilon)$[38] and then for $q \geqq n_1 = n_5$, $\mathbf{N}_1 = \mathbf{N}_1(q, A_1, \cdots, A_m, \epsilon) = \mathbf{O}$.[38] Then our $C_1, \cdots, C_m$ replace the $B_1, \cdots, B_m$ of Def. 4.3.1, and (i)–(iii) there obtain as follows: (i) holds by virtue of the construction of $\mathbf{O}$ (ii) follows from (4.6.ρ), and (iii) coincides with (4.6.υ).

Thus the proof is completed.

We could now make the final steps as indicated after Lemma 4.5.1 above, but we introduce one more variant of the concept of approximate finiteness:

DEFINITION 4.6.1. $\mathbf{M}$ *is approximately finite* (C) *if*:

There exists a sequence of rings $\mathbf{N}_1, \mathbf{N}_2, \cdots$ *with the following properties*:

(i) $\mathbf{N}_n$ *is of finite order*

(ii) $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots$

(iii) *The ring* $\mathbf{M}$ *is generated by the* $\mathbf{N}_1, \mathbf{N}_2, \cdots$

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n ; n = 1, 2, \cdots).$$

---

[38] Observe that the (indirectly defined) expression at the extreme right has the proper dependence—i.e. that one indicated by the expression in the middle.

LEMMA 4.6.3. *If* $\mathbf{M}$ *is approximately finite of type* $[p_1, p_2, \cdots]$ *then it is also approximately finite* (C).

PROOF. Immediate by comparison of Defs. 4.1.1 and 4.6.1.

LEMMA 4.6.4. *If* $\mathbf{M}$ *is approximately finite* (C), *then it is also approximately finite* (B).

PROOF. (ii) and (iii) in Def. 4.6.1 coincide with those in Def. 4.1.1. Hence Lemma 4.1.4 applies to our $\mathbf{N}_1, \mathbf{N}_2, \cdots$.

Let then the $A_1, \cdots, A_m \in \mathbf{M}$ and the $\epsilon > 0$ of Def. 4.5.2 be given. Owing to Lemma 4.1.4, A_k is the limit of a metrically convergent sequence from the set-theoretical sum of the $\mathbf{N}_1, \mathbf{N}_2, \cdots$. Hence there exists a $B_k \in \mathbf{N}_{n_k}$ with

$$[[B_k - A_k]] < \epsilon.$$

Now put $n = \text{Max}\ (n_1, \cdots, n_m)$. Then every $B_k \in \mathbf{N}_{n_k} \subseteq \mathbf{N}_n$. Thus $\mathbf{N} = \mathbf{N}_n$ meets all the requirements (i)–(iii) of Def. 4.5.2.

We can now prove

THEOREM XII. *All kinds of approximate finiteness are equivalent to each other*: (A), (B), (C) *and every type* $[p_1, p_2, \cdots]$ (*for every sequence* $[p_1, p_2, \cdots]$ *which fulfills* (i), (ii) *of Lemma* 4.1.3).

We shall therefore use the expression *approximately finite* from now on, without any further qualification.

PROOF. The implications

$$(A) \rightarrow \text{type } [p_1, p_2, \cdots] \rightarrow (B) \rightarrow (A)$$

were established in Lemmas 4.4.3, 4.5.1 and 4.6.2 respectively. Consequently

$$\text{type } [p_1, p_2, \cdots] \leftrightarrows (A) \leftrightarrows (B).$$

Further

$$\text{type } [p_1, p_2, \cdots] \rightarrow (C) \rightarrow (B)$$

by Lemmas 4.6.3, 4.6.4 respectively; hence

$$\text{type } [p_1, p_2, \cdots] \leftrightarrows (C)$$

too.

§4.7 We have not shown so far that approximately finite factors exist at all. In §§5.2, 5.3 and 5.6 we shall give various examples of such factors. Actually it is more difficult to show that there are factors in case (II_1) which are not approximately finite.

Our immediate goal is to prove a general theorem which yields the existence of approximately finite factors as a byproduct.

A preparatory lemma is necessary:

LEMMA 4.7.1. *Let a factor* $\mathbf{N} \subseteq \mathbf{M}$ *in a case* (I_p), $p = 1, 2, \cdots$ *be given. If an* $A \in \mathbf{N}$ *possesses these two properties*

(i) $[[AX - XA]] \leqq \epsilon \,|||\, X \,|||$ *for every* $X \in \mathbf{N}$

(ii) $Tr_{\mathbf{M}}(A) = 0$,

then $[[A]] \leqq \epsilon$.

PROOF. For any unitary $U \epsilon \mathbf{N}$ we have $|||\, U \,||| = |||\, U^{-1} \,||| = 1$. Hence by (i),

$$[[UAU^{-1} - A]] = [[U(AU^{-1} - U^{-1})A]] \leqq [[AU^{-1} - U^{-1}A]] \leqq \epsilon.$$

Also by (ii)

$$Tr_{\mathbf{M}}(UAU^{-1}) = Tr_{\mathbf{M}}(A) = 0.$$

If $U_1, \cdots, U_m$ are unitary and $\epsilon \mathbf{N}$ then we have by the above $[[U_iAU_i^{-1} - A]] \leqq \epsilon$, $Tr_{\mathbf{M}}(U_iAU_i^{-1}) = 0$. So for

$$B = (1/m)(\textstyle\sum_{i=1}^{m} U_i A U_i^{-1}) \tag{4.7.α}$$

the evaluations

$$[[B - A]] \leqq \epsilon \tag{4.7.β}$$

$$Tr_{\mathbf{M}}(B) = 0 \tag{4.7.γ}$$

obtain.

Now choose an isomorphism $\mathfrak{K}$ between $\mathbf{N}$ and the system of all (complex) p-order matrices, in the sense of (α) in Lemma 4.1.1. Put

$$\mathfrak{K}A = a = \{a_{t,s}\}, \qquad \mathfrak{K}B = b = \{b_{t,s}\}.$$

Form next the operators $V, X_1, \cdots, X_p \epsilon \mathbf{N}$ with

$$\mathfrak{K}V = v = \{v_{s,t}\},\ v_{s,t} \begin{cases} = 1 & \text{if } s - t \equiv 1 \bmod p \\ = 0 & \text{otherwise} \end{cases}$$

$$\mathfrak{K}X = x_r = \{x^r_{s,t}\},\ x^r_{s,t} \begin{cases} = 1 & \text{if } t = s \neq r \\ = -1 & \text{if } t = s = r \\ = 0 & \text{otherwise.} \end{cases}$$

The matrices v, x_r are unitary. Hence the operators V, X_r are also unitary.

Now put $m = p2^p$ and let $U_1, \cdots, U_m$ be the operators $V^hX_1^{k_1} \cdot \cdots \cdot X_p^{k_p}$, $h = 0, 1, \cdots, p-1; k_1, \cdots, k_p = 0, 1$. Then a combination of these formulas gives, due to (4.7.α) by direct computation,[39]

$$b_{t,s} \begin{cases} = (1/p) \textstyle\sum_{u=1}^{p} a_{u,u} = a_0 & \text{for } t = s \\ = 0 & \text{otherwise.} \end{cases}$$

Consequently

$$B = a_0 \cdot 1 \tag{4.7.δ}$$

---

[39] This computation may be outlined as follows. Let $\epsilon^{(r)}_{s,t} = (-1)^{\delta_{r,s}+\delta_{r,t}}$, i.e.

$$\epsilon^{(r)}_{s,t} \begin{cases} = 1 \text{ if } r \neq t \text{ and } s \neq r \text{ or if } r = s = t \\ = -1 \text{ if either } r = s \text{ and } r \neq t \text{ or } r \neq s \text{ and } r = t \end{cases}$$

Then if $\mathfrak{K}X_rAX_r^{-1} = c^{(r)} = (c^{(r)}_{s,t})$

$$c^{(r)}_{s,t} = \epsilon^r_{s,t}\, a_{s,t}\,.$$

(*continued*)

(4.7.δ) and (4.7.γ) give together $0 = Tr_{\mathbf{M}}(B) = a_0$, i.e. $B = 0$. Hence (4.7.β) becomes $[[A]] \leqq \epsilon$ as desired.

We are able to prove:

THEOREM XIII. *If* $\mathbf{M}$ *is a factor in case* (II_1), $\mathbf{M}$ *contains a subring* $\mathbf{O}$ *which is a factor in case* (II_1), *and approximately finite.*

PROOF. Choose a sequence $[p_1, p_2, \cdots]$ which fulfills (i), (ii) in Lemma 4.1.3, e.g. $p_n = 2^n$.

Put $p_0 = 1$ and $\mathbf{N}_0 = (\alpha 1)$. This is a factor $\subsetneqq \mathbf{M}$ and in case (I_1). We shall choose by induction for every $n = 1, 2, \cdots$ a factor $\mathbf{N}_n \subsetneqq \mathbf{M}$ in the case (I_{p_n}). If for any $n = 1, 2, \cdots, \mathbf{N}_{n-1}$ has already been chosen, we choose $\mathbf{N}_n$ in the following way: $\mathbf{N}_{n-1}$, $\mathbf{M}$ are factors in the cases ($\mathrm{I}_{p_{n-1}}$), (II_1) respectively; $\mathbf{N}_{n-1} \subsetneqq \mathbf{M}$, p_{n-1} is a divisor of p_n. Apply therefore Lemma 4.4.2 to $\mathbf{N}_{n-1}$, $\mathbf{M}$, p_{n-1}, p_n in place of its $\mathbf{N}$, $\mathbf{O}$, p, r. Put $\mathbf{N}_n = \mathbf{R}$. Then $\mathbf{N}_n$ is a factor $\subsetneqq \mathbf{M}$ in the case (I_{p_n}).

Thus

$$\mathbf{N}_1 \subsetneqq \mathbf{N}_2 \subsetneqq \cdots \subsetneqq \mathbf{M}. \tag{4.7.ϵ}$$

Denote by $\mathbf{S}$ the set-theoretical sum of the sequence $\mathbf{N}_1, \mathbf{N}_2, \cdots$. Owing to (4.7.$\epsilon$), $\mathbf{S}$ is an algebra along with $\mathbf{N}_1, \mathbf{N}_2, \cdots$.

Denote the set of the limits of all metrically convergent sequences from $\mathbf{S}$ by $\mathbf{S}_1$. The same argument as in the proof of Lemma 4.1.4 shows that $\mathbf{S}_1$ is an algebra along with $\mathbf{S}$ and then that it is a ring. Hence $\mathbf{S} \subseteq \mathbf{S}_1$ implies $\mathbf{R}(\mathbf{S}) \subseteq \mathbf{S}_1$. On the other hand $\mathbf{R}(\mathbf{S})$ is closed in the sense of metric convergence by Theorem I. Hence $\mathbf{S} \subseteq \mathbf{R}(\mathbf{S})$ implies $\mathbf{S}_1 \subseteq \mathbf{R}(\mathbf{S})$. Consequently $\mathbf{R}(\mathbf{S}) = \mathbf{S}_1$ and so

$$\mathbf{M} \supseteq \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) = \mathbf{R}(\mathbf{S}) = \mathbf{S}_1. \tag{4.7.ζ}$$

Next we prove that the ring $\mathbf{S}_1$ is a factor, i.e. that every $A \in \mathbf{S}_1 \cdot \mathbf{S}_1'$ belongs to $(\alpha \cdot 1)$.

Let $B_0 = \sum_{k_1,\cdots,k_p=0,1} (X_1^{k_1} \cdot \cdots \cdot X_p^{k_p}) A (X_1^{k_1} \cdot \cdots \cdot X_{k_p}^{k_p})^{-1}$ and $\mathfrak{K} B_0 = d = (d_{t,s})$. Then

$$\begin{aligned} d_{t,s} &= \sum_{k_1,\cdots,k_p=0,1} (\epsilon_{s,t}^{(1)})^{k_1} \cdot \cdots \cdot (\epsilon_{s,t}^{(p)})^{k_p} a_{s,t} \\ &= ((\epsilon_{s,t}^{(1)})^0 + (\epsilon_{s,t}^{(1)})^1) \cdot \cdots \cdot ((\epsilon_{s,t}^{(p)})^0 + (\epsilon_{s,t}^{(p)})^1) a_{s,t} \\ &= (1 + \epsilon_{s,t}^{(1)}) \cdot \cdots \cdot (1 + \epsilon_{s,t}^{(p)}) a_{s,t}. \end{aligned}$$

Unless $s = t$, $\epsilon_{s,t}^{(\nu)} = -1$ and $d_{s,t} = 0$. If $s = t$, $d_{s,s} = 2^p a_{s,s}$. Thus B_0 is a diagonal matrix.

The effect of V on a diagonal matrix is just to permute the diagonal terms cyclicly. Hence if B and $b_{s,t}$ are as in the text

$$B = (1/p2^p) \sum_{h=0}^{p-1} V^h B_0 V^{-h}$$

and

$$\begin{aligned} b_{s,t} &= 0 \quad \text{if} \quad s \neq t \\ b_{s,s} &= (1/p) \sum_{u=1}^{p} a_{u,u}. \end{aligned}$$

Consider an $A \in \mathbf{S}_1 \cdot \mathbf{S}_1'$. Choose $\epsilon > 0$. As $A \in \mathbf{S}_1$ there exists a $B_\epsilon \in \mathbf{N}_{p_{n_\epsilon}}$ with

$$[[B_\epsilon - A]] < \epsilon. \tag{4.7.η}$$

Put $C_\epsilon = B_\epsilon - Tr_{\mathbf{M}}(B_\epsilon) \cdot 1$. Then $C_\epsilon \in \mathbf{N}_{p_{n_\epsilon}}$ too, and

$$Tr_{\mathbf{M}}(C_\epsilon) = 0. \tag{4.7.θ}$$

Consider an $X \in \mathbf{N}_{p_{n_\epsilon}}$. As $A \in \mathbf{S}_1'$, $X \in \mathbf{S}_1$, so $AX - XA = 0$. Also, clearly $B_\epsilon X - XB_\epsilon = C_\epsilon X - XC_\epsilon$. Hence $C_\epsilon X - XC_\epsilon = B_\epsilon X - XB_\epsilon = (B_\epsilon - A)X - X(B_\epsilon - A)$ and so $[[C_\epsilon X - XC_\epsilon]] \leqq 2 \,|||\, X \,|||\, [[B_\epsilon - A]]$. This and (4.7.$\eta$) yield

$$[[C_\epsilon X - XC_\epsilon]] \leqq 2 \,|||\, X \,|||\, \cdot \epsilon. \tag{4.7.ι}$$

(4.7.θ) and (4.7.ι) permit us to apply Lemma 4.7.1 to C_ϵ, 2ϵ in place of its A, ϵ. Thus

$$[[C_\epsilon]] \leqq 2\epsilon.$$

Put $Tr_{\mathbf{M}}(B_\epsilon) = \alpha_\epsilon$. Then the above equation means

$$[[B_\epsilon - \alpha_\epsilon \cdot 1]] \leqq 2\epsilon.$$

Hence by (4.7.η)

$$[[A - \alpha_\epsilon \cdot 1]] < 3\epsilon. \tag{4.7.κ}$$

As (4.7.κ) holds for all $\epsilon > 0$, it proves that A is the limit of a metrically convergent sequence from $(\alpha \cdot 1)$. As $(\alpha \cdot 1)$ is obviously closed in this sense, we have $A \in (\alpha \cdot 1)$ as desired.

Thus $\mathbf{S}_1$ is a factor. The considerations at the beginning of the proof of (i) in Lemma 4.1.2 show again that $D_{\mathbf{S}_1}(E) = D_{\mathbf{M}}(E)$ for all projections $E \in \mathbf{S}_1$. Hence $\mathbf{S}_1$ is in a finite case, along with $\mathbf{M}$. If $\mathbf{S}_1$ were in a case (I_q), $q = 1, 2, \cdots$, we could argue as follows: $\mathbf{N}_{p_n} \subsetneqq \mathbf{S}_1$ and $\mathbf{N}_{p_n}$ is a factor in case (I_{p_n}). Hence p_n is a divisor of q by (i) in Lemma 4.1.2. But this is impossible, since $p_n \to \infty$ as $n \to \infty$. Therefore $\mathbf{S}_1$ must be in the case (II_1).

Now (4.7.ϵ), (4.7.ζ) prove that $\mathbf{S}_1$ is approximately finite of type $[p_1, p_2, \cdots]$—i.e. approximately finite by Theorem XII.

Thus $\mathbf{O} = \mathbf{S}_1$ meets all our requirements.

And to conclude:

THEOREM XIV. *Approximately finite factors* $\mathbf{M}$ *(in the case* II_1*) exist and they have all the same algebraical type* $\bar{\mathbf{M}}$.

We denote this algebraical type by $\bar{\mathbf{A}}$.

PROOF. The existence follows from Theorem XIII.

Consider two approximately finite $\mathbf{M}_1$, $\mathbf{M}_2$. Choose a sequence $[p_1, p_2, \cdots]$ which fulfills (i), (ii) in Lemma 4.1.3, e.g. $p_n = 2^n$. Then $\mathbf{M}_1$, $\mathbf{M}_2$ are both approximately finite of type $[p_1, p_2, \cdots]$ by Theorem XII. Hence they are algebraically isomorphic by Theorem XI, i.e. $\bar{\mathbf{M}}_1 = \bar{\mathbf{M}}_2$.

Thus $\bar{\mathbf{M}}$ ($\mathbf{M}$ approximately finite) is uniquely determined and the definition of $\bar{\mathbf{A}}$ is justified.

§4.8 The algebraical isomorphism invariants described at the end of §2.10 will now be studied for the algebraical type $\bar{\mathbf{A}}$. These invariants are

(1) Validity of $\bar{\mathbf{A}}^c = \bar{\mathbf{A}}$

(2) The fundamental group: $\mathfrak{G} = \mathfrak{G}(\bar{\mathbf{A}})$.

LEMMA 4.8.1. $\bar{\mathbf{A}}^c = \bar{\mathbf{A}}$.

PROOF. By Theorem XIV on one hand and Theorem III on the other, it suffices to show that a definition of approximate finiteness is unaffected by a conjugate or by a dual isomorphism. This is obviously true, even for all the definitions given in Theorem XII and for both types of general isomorphisms (cf. B^*) in §2.2) mentioned above.

LEMMA 4.8.2. $\mathfrak{G} = \mathfrak{G}(\bar{\mathbf{A}})$ *contains all rational numbers* θ *in* $0 < \theta < \infty$.

PROOF. Since $\mathfrak{G}$ is a group, it suffices to prove that it contains all $1/p$, $p = 2, 3, \cdots$.

Consider such a $1/p$. Choose a sequence $[p_1, p_2, \cdots]$ which fulfills (i), (ii) in Lemma 4.1.3 and with all p_m divisible by p, e.g. $p_n = p^n$.

Choose an approximately finite $\mathbf{M}$ and the $\mathbf{N}_1, \mathbf{N}_2, \cdots$, for this $\mathbf{M}$ and the sequence $[p_1, p_2, \cdots]$ in the sense of Def. 4.1.1.

The considerations at the beginning of the proof of (i) in Lemma 4.1.2 show again that $D_{\mathbf{M}}(G) = D_{\mathbf{N}_n}(G)$ for all projections $G \in \mathbf{N}_n (\subseteqq \mathbf{M})$.

As $\mathbf{N}_1$ is in the case (I_{p_1}), p_1 divisible by p, there exists a projection $E \in \mathbf{N}_1$ with $D_{\mathbf{N}_1}(E) = 1/p$. Thus

$$E \in \mathbf{N}_1 \subseteqq \mathbf{N}_2 \subseteqq \cdots \subseteqq \mathbf{M} \tag{4.8.α}$$

while $D_{\mathbf{N}_1}(E) = 1/p$ yields

$$D_{\mathbf{O}}(E)/D_{\mathbf{O}}(1) = D_{\mathbf{O}}(E) = 1/p \quad \text{for} \quad \mathbf{O} = \mathbf{N}_1, \mathbf{N}_2, \cdots, \mathbf{M}. \tag{4.8.β}$$

Denote the range of E by $\mathfrak{M}$.

By (4.8.α) we can form the factors $\mathbf{N}_{1(\mathfrak{M})}, \mathbf{N}_{2(\mathfrak{M})}, \cdots \mathbf{M}_{(\mathfrak{M})}$ in $\mathfrak{M}$, and we have

$$\mathbf{N}_{1(\mathfrak{M})} \subseteqq \mathbf{N}_{2(\mathfrak{M})} \subseteqq \cdots \subseteqq \mathbf{M}_{(\mathfrak{M})}. \tag{4.8.γ}$$

By (4.8.β), $\overline{\mathbf{N}_{n(\mathfrak{M})}} = \bar{\mathbf{N}}_n^{1/p}$, $\overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{M}}^{1/p}$ (cf. the discussion before Theorem VI). The former implies by Lemma 2.10.3 that

$$\mathbf{N}_{n(\mathfrak{M})} \text{ is in the case } (\mathrm{I}_{p_n/p}). \tag{4.8.δ}$$

We restate the latter:

$$\overline{\mathbf{M}_{(\mathfrak{M})}} = \bar{\mathbf{M}}^{1/p}. \tag{4.8.ϵ}$$

We know that $\mathbf{M}$ is the closure of the set-theoretical sum of $\mathbf{N}_1, \mathbf{N}_2, \cdots$ in the strong topology.[40] Hence it follows from (4.8.α) that $\mathbf{M}_{(\mathfrak{M})}$ is the closure of

[40] That closure is obviously $= \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots)$. Hence it is $\mathbf{M}$ by (iii) in Def. 4.1.1.

the set-theoretical sum of $\mathbf{N}_{1(\mathfrak{M})}$, $\mathbf{N}_{2(\mathfrak{M})}$, $\cdots$ in the strong topology. This and (4.8.γ) give together

$$\mathbf{M}_{(\mathfrak{M})} = \mathbf{R}(\mathbf{N}_{n(\mathfrak{M})},\ n = 1, 2, \cdots). \tag{4.8.ζ}$$

Now (4.8.δ), (4.8.γ) and (4.8.ζ) coincide with (i), (ii), (iii) in Def. 4.1.1 as applied to $\mathbf{M}_{(\mathfrak{M})}$, $\mathbf{N}_{1(\mathfrak{M})}$, $\mathbf{N}_{2(\mathfrak{M})}$, $\cdots$, p_1/p, p_2/p_2, $\cdots$ in place of its $\mathbf{M}$, $\mathbf{N}_1$, $\mathbf{N}_2$ $\cdots$, p_1, p_2, $\cdots$. Hence $\mathbf{M}_{(\mathfrak{M})}$ is approximately finite.

As $\mathbf{M}$ is approximately finite too, this shows that $\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}}$ by Theorem XIV. Owing to (4.8.ϵ), this gives $\overline{\mathbf{M}}^{1/p} = \overline{\mathbf{M}}$. Hence $1/p \in \mathfrak{G}$ as desired.

LEMMA 4.8.3. *If there exists for every $\epsilon > 0$ an α in $1 - \epsilon < \alpha < 1$ such that $\overline{\mathbf{M}}^{\alpha}$ is approximately finite, then $\mathbf{M}$ itself is approximately finite.*

PROOF. Assume that $\mathbf{M}$ possesses the property described above. We shall establish its approximate finiteness in the sense of Def. 4.5.2.

Let accordingly $A_1, \cdots, A_m \in \mathbf{M}$ and an $\epsilon > 0$ be given. We shall construct an $\mathbf{N}$ which fulfills (i)–(iii) in that definition.

Apply first our hypothesis concerning $\mathbf{M}$ to

$$\epsilon' = (\epsilon/4 \operatorname{Max}(|||A_s|||, s = 1, \cdots, m))^2 \tag{4.8.η}$$

in place of its ϵ. Choose the α mentioned there with $1 - \epsilon' < \alpha < 1$. Then choose a projection $E \in \mathbf{M}$ with $D_{\mathbf{M}}(E) = \alpha$. Let $\mathfrak{M}$ be the range of E. Form $\mathbf{M}_{(\mathfrak{M})}$. Then we have $\overline{\mathbf{M}_{(\mathfrak{M})}} = \overline{\mathbf{M}}^{\alpha}$ (cf. the discussion before Theorem VI). It follows from our assumptions concerning α that $\mathbf{M}_{(\mathfrak{M})}$ is approximately finite.

By [5], p. 187, (i), in Lemma 11.3.3,

$$A \leftrightarrows A_{(\mathfrak{M})}$$

is a one-to-one correspondence between all $A \in \mathbf{M}$ with $EA = AE = A$ and all $A_{(\mathfrak{M})} \in \mathbf{M}_{(\mathfrak{M})}$. Now we can repeat with a slight variation an argument that was used in the proof of Theorem XI: Clearly $cTr_{\mathbf{M}}(A)$ will do as $Tr_{\mathbf{M}_{(\mathfrak{M})}}(A_{(\mathfrak{M})})$ in the sense of [6], p. 219, Property IV, for the A, $A_{(\mathfrak{M})}$ described above, if we choose the normalizing factor $c > 0$ so that $cTr_{\mathbf{M}}(A) = 1$ when $A_{(\mathfrak{M})}$ is the unit of $\mathbf{M}_{(\mathfrak{M})}$. This means $A = E$, so we want $cTr_{\mathbf{M}}(E) = 1$. Since $Tr_{\mathbf{M}}(E) = D_{\mathbf{M}}(E) = \alpha$ this means $c = 1/\alpha$. And as the trace is uniquely characterized by the above property, $Tr_{\mathbf{M}_{(\mathfrak{M})}}(A_{(\mathfrak{M})}) = (1/\alpha)Tr_{\mathbf{M}}(A)$. From this $(1/\alpha)Tr_{\mathbf{M}}(A^*A) = Tr_{\mathbf{M}_{(\mathfrak{M})}}((A^*A)_{(\mathfrak{M})}) = Tr_{\mathbf{M}_{(\mathfrak{M})}}((A^*)_{(\mathfrak{M})} \cdot (A)_{(\mathfrak{M})})$ for the same A, $A_{(\mathfrak{M})}$ hence

$$[[A_{(\mathfrak{M})}]]_{\mathbf{M}_{(\mathfrak{M})}} = (1/\alpha)^{\frac{1}{2}}[[A]]_{\mathbf{M}}.^{41} \tag{4.8.θ}$$

Now put $B_s = EA_sE$, $s = 1, \cdots, m$. Then we have $D_{\mathbf{M}}(1 - E) < \epsilon'$. Hence $[[1 - E]]_{\mathbf{M}} < \epsilon'^{\frac{1}{2}}$ [35] and therefore $[[A_s - B_s]]_{\mathbf{M}} = [[A_s(1 - E) + (1 - E)A_sE]] < |||A_s||| \epsilon'$. Hence by (4.8.$\eta$)

$$[[A_s - B_s]]_{\mathbf{M}} < \epsilon/2. \tag{4.8.ι}$$

[41] As a rule we did not indicate the factor (in a finite case) with respect to which $[[A]]$ is formed. However, here it is desirable.

Now $B_s \epsilon \mathbf{M}$ and $EB_s = B_sE = B_s$ imply $B_{s(\mathfrak{M})} \epsilon \mathbf{M}_{(\mathfrak{M})}$. We now apply Def. 4.5.2 to the approximately finite $\mathbf{M}_{(\mathfrak{M})}$ with $B_{1(\mathfrak{M})}, \cdots, B_{m(\mathfrak{M})}, \epsilon/2\alpha^{\frac{1}{2}}$ in place of its $A_1, \cdots, A_m$ and ϵ. This yields a ring $\mathbf{N}^+ \subseteq \mathbf{M}_{(\mathfrak{M})}$ of finite order with $C_1^+, \cdots, C_m^+ \epsilon \mathbf{N}^+$ such that

$$[[C_s^+ - B_{s(\mathfrak{M})}]]_{\mathbf{M}_{(\mathfrak{M})}} < \epsilon/2\alpha^{\frac{1}{2}}. \tag{4.8.κ}$$

As $\mathbf{N}^+ \subseteq \mathbf{M}_{(\mathfrak{M})}$ we extend every $C^+ \epsilon \mathbf{N}^+$ to a $C \epsilon \mathbf{M}$ with $EC = CE = C$ and $C^+ = C_{(\mathfrak{M})}$.[42] These C form obviously a ring $\mathbf{N} \subseteq \mathbf{M}$ of finite order with $\mathbf{N}^+ = \mathbf{N}_{(\mathfrak{M})}$. By this process the $C_1^+, \cdots, C_m^+ \epsilon \mathbf{N}^+$ give rise to the $C_1, \cdots, C_m \epsilon \mathbf{N}$. Now (4.8.$\theta$) gives

$$[[C_s^+ - B_{s(\mathfrak{M})}]]_{\mathbf{M}_{(\mathfrak{M})}} = [[(C_s - B_s)_{(\mathfrak{M})}]]_{\mathbf{M}_{(\mathfrak{M})}} = (1/\alpha^{\frac{1}{2}})[[C_s - B_s]]_{\mathbf{M}}$$

and so (4.8.κ) becomes

$$[[C_s - B_s]]_{\mathbf{M}} < \epsilon/2. \tag{4.8.λ}$$

(4.8.ι) and (4.8.λ) together give

$$[[C_s - A_s]]_{\mathbf{M}} < \epsilon. \tag{4.8.μ}$$

Thus the ring $\mathbf{N} \subseteq \mathbf{M}$ of finite order meets the requirements (i)–(iii) in Def. 4.5.2. Hence $\mathbf{M}$ is approximately finite as asserted.

LEMMA 4.8.4. *$\mathfrak{G} = \mathfrak{G}(\bar{\mathbf{A}})$ is the set P of all (real) numbers θ in $0 < \theta < \infty$.*

PROOF. Consider a θ in $0 < \theta < \infty$. By Lemma 4.8.2 we have for every rational ξ in $0 < \xi < \infty$, $\bar{\mathbf{A}}^\xi = \bar{\mathbf{A}}$. Hence $\bar{\mathbf{A}}^\xi$ is approximately finite. Therefore $(\bar{\mathbf{A}}^\theta)^\alpha = \bar{\mathbf{A}}^{\theta\alpha}$ is approximately finite if $\theta\alpha$ is rational.

Now it is obviously possible to find for any given $\epsilon > 0$ an α in $1 - \epsilon < \alpha < 1$ for which $\theta\alpha$ is rational. This makes $(\bar{\mathbf{A}}^\theta)^\alpha$ approximately finite. Hence $\bar{\mathbf{A}}^\theta$ itself is approximately finite by Lemma 4.8.3 with $\bar{\mathbf{A}}^\theta$ in place of its $\mathbf{M}$. Consequently Theorem XIV gives $\bar{\mathbf{A}}^\theta = \mathbf{A}$. Hence $\theta \epsilon \mathfrak{G}$ as desired.

We restate the results of Lemmas 4.8.1 and 4.8.4.

THEOREM XV. *The algebraical isomorphism invariants in* (1), (2) *at the beginning of this section, behave for the approximate finite type $\bar{\mathbf{A}}$ as follows*:

(1) $\bar{\mathbf{A}}^c = \bar{\mathbf{A}}$

(2) $\mathfrak{G}(\bar{\mathbf{A}}) = P$.

It is worth pointing out that the two invariants (1), (2) are not sufficient to characterize an algebraical type $\bar{\mathbf{M}}$ in the case (II_1). A proof of this is only possible if we show first that not all $\bar{\mathbf{M}}$ are approximately finite: If all were, then Theorem XIV would give $\bar{\mathbf{M}} = \bar{\mathbf{A}}$, i.e. only one $\bar{\mathbf{M}}$ in case (II_1) would exist and thus no invariants for these $\bar{\mathbf{M}}$ would be needed at all.

Now we shall prove in Theorems XVI, XVI′ that not approximately finite $\bar{\mathbf{M}}$ in the case (II_1) exist. By making advance use of this result we prove:

LEMMA 4.8.5. *The algebraical isomorphism invariants in* (1), (2) *at the beginning of this paragraph are not sufficient to characterize the algebraical type $\bar{\mathbf{M}}$ in the case* (II_1).

[42] C is reduced by $\mathfrak{M}$, its part in $\mathfrak{M}$ is C^+ and its part in $\mathfrak{M}^-$ is 0.

PROOF. Assume the opposite.

Consider a type $\overline{\mathbf{M}}$ in the case (II_1).

Owing to the formulae (2.3.α) and (2.8.β) the validity of $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$ is unaffected if we replace $\overline{\mathbf{M}}$ by $\overline{\mathbf{M}}^c$ or by an $\overline{\mathbf{M}}^\theta (0 < \theta < \infty)$. By Lemma 2.10.2 $\mathfrak{G}(\overline{\mathbf{M}})$ is unchanged if we replace $\overline{\mathbf{M}}$ by $\overline{\mathbf{M}}^c$ or by an $\overline{\mathbf{M}}^\theta (0 < \theta < \infty)$. In other words: The invariants (1) and (2) are unaffected by a replacement of $\overline{\mathbf{M}}$ by $\overline{\mathbf{M}}^c$ or by an $\overline{\mathbf{M}}^\theta$.

Therefore our assumptions concerning these invariants permit us to conclude $\overline{\mathbf{M}} = \overline{\mathbf{M}}^c$ and $\overline{\mathbf{M}} = \overline{\mathbf{M}}^\theta$. The latter equations imply $\theta \in \mathfrak{G}(\overline{\mathbf{M}})$ and therefore $\mathfrak{G}(\overline{\mathbf{M}}) = P$.

Thus the two invariants (1) and (2) are uniquely determined, independently of the choice of $\overline{\mathbf{M}}$ in the case (II_1).

Hence a second application of our assumption concerning these invariants yields that there exists only one type $\overline{\mathbf{M}}$ in the case (II_1). $\overline{\mathbf{A}}$ is such a type; hence all $\overline{\mathbf{M}}$ in the case (II_1) must equal $\overline{\mathbf{A}}$, i.e. be approximately finite.

However, as stated above, the opposite will be proven in Theorems XVI, XVI′, and we propose to use this now. Thus we have a contradiction and the proof is completed.

We conclude with the observation that the existence of not approximately finite factors $\mathbf{M}$ in the case (II_1) will be established with the help of algebraic invariants other than (1), (2) above (cf. the beginning of §6.1).

CHAPTER V. ON VARIOUS EXAMPLES

§5.1 We established the existence of approximately finite factors (i.e. of their type $\overline{\mathbf{A}}$) by the indirect procedure of Theorems XIII and XIV.[43] It is therefore desirable to furnish examples of this by explicit and direct construction and also to investigate those examples of factors in the case (II_1) which we possess already as to their approximate finiteness. The latter investigation is also necessary in order to find not approximately finite factors in the case (II_1).

In connection with this work we shall develop a new technique to construct examples of factors in the case (II_1), which is a simplification of the original one of [5], pp. 192–209, Part IV.

We begin by establishing the approximate finiteness of some of our old examples: certain ones in [5] and that one in [7], pp. 72–77, §7.5.

§5.2 LEMMA 5.2.1. *The ring* $\mathbf{C}^{*1}$ *of* [7], *p.* 72, *Lemma* 7.5.1 *is approximately finite.*

PROOF. Using the notations loc. cit., particularly pp. 68, 72, we have

$$\mathbf{C}^{*1} = \mathbf{R}(\overline{\overline{\mathbf{B}}}_{(n,1)} ; n = 1, 2, \cdots).$$

Hence

$$\mathbf{C}^{*1} = \mathbf{R}(\mathbf{N}_m ; m = 1, 2, \cdots) \tag{5.2.α}$$

[43] The decisive passage from Theorem XIII to Theorem XIV makes use of the existence of factors in the case (II_1), for instance the examples of [5].

with

$$(5.2.\beta) \qquad \mathbf{N}_m = \mathbf{R}(\bar{\bar{\mathbf{B}}}_{(n,1)}\ ;\ n = 1, \cdots, m).$$

Thus $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots$ and therefore $\mathbf{C}^{*1}$ is approximately finite by Def. 4.6.1 if the $\mathbf{N}_m$ are shown to be of finite order. [44]

These considerations belong in $\prod^{\mathfrak{D}}_{\otimes,n=1,2,\cdots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ where $\mathbf{C}^{*1}$ is a factor in the case (II_1), but in establishing the finite order of $\mathbf{N}_m$, we may as well consider the entire space $\prod_{\otimes,n=1,2,\cdots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$.

We proceed as in the lower part of p. 68, loc. cit. $\mathbf{B}_{(n,1)}$ is in $\mathfrak{H}_{(n,1)}$. It is first extended to $\bar{\mathbf{B}}_{(n,1)}$ in $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ and then to $\bar{\bar{\mathbf{B}}}_{(n,1)}$ in $\prod_{\otimes,n=1,2,\cdots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$. Hence the $\bar{\bar{\mathbf{B}}}_{(m,1)}$ commute with each other. Therefore the $\mathbf{N}_m$ of $(5.2.\beta)$ is of finite order if the $\bar{\bar{\mathbf{B}}}_{(n,1)}$ are. Now $\mathbf{B}_{(n,1)}$ is of finite order because it is in the finite 2-dimensional space $\mathfrak{H}_{(n,1)}$ and so $\bar{\bar{\mathbf{B}}}_{(n,1)}$ is of finite order too.

LEMMA 5.2.2. *The ring* $\mathbf{M}$ *of* [5], *pp.* 203–204, *Theorem XI, is approximately finite if the group* $\mathfrak{G}$ *possesses this property*:

(i) $\mathfrak{G}$ *is the set-theoretical sum of a sequence* $\mathfrak{G}_1 \subseteq \mathfrak{G}_2 \subseteq \cdots$ *of finite groups.*

(We assume, of course, that the requirements of loc. cit. [5], p. 206, Lemma 13.1.2, are fulfilled, placing $\mathbf{M}$ in the case (II_1).)

PROOF. Using the notations loc. cit., particularly those on pp. 198–200, we have

$$(5.2.\gamma) \qquad \mathbf{M} = \mathbf{R}(\bar{U}_{a_0}, \bar{L}_{\varphi(x)};\ a \in \mathfrak{G},\ \varphi(x) \text{ bounded and measurable}).$$

A familiar argument shows that the $\varphi(x)$ in $(5.2.\gamma)$ can be restricted considerably without changing the ring $\mathbf{M}$. Indeed:

First. It suffices for reasons of continuity to consider the $\varphi(x)$ with finite ranges only.

Second. These $\varphi(x)$ are finite linear combinations of the characteristic functions

$$\chi_S(x) \begin{cases} = 1 & \text{for } x \in S \\ = 0 & \text{otherwise.} \end{cases}$$

Third. Let $T_1, T_2, \cdots$ be a countable basis for the T ($\subseteq S$ and measurable). [45]

---

[44] It is not difficult to show that $\mathbf{N}_n$ is a factor in the case (I_{2^n}). Hence Def. 4.1.1. would also be applicable, yielding approximate finiteness of type [2, 4, 8, $\cdots$].

[45] The space of all measurable sets T (disregarding sets of measure zero) can be metricized by the distance

$$\overline{T'T''} = \mu(T' + T'') - \mu(T'T'').$$

This is obviously equal to

$$|\chi_{T'} - \chi_{T''}|^2 = \int_S |\chi_{T'}(x) - \chi_{T''}(x)|^2\, dx.$$

Since $\mathfrak{H}_S$ is separable, there exists a sequence $\chi_{T_1}, \chi_{T_2}, \cdots$ dense in the set of all χ_T, and correspondingly the sequence $T_1, T_2, \cdots$ is dense in the set of all T. This is what we mean by a basis.

Then it suffices for reasons of continuity to consider these $\varphi(x) = \chi_{T_i}(x)$, $i = 1, 2, \cdots$ only.

Finally, these properties of the system $T_1, T_2, \cdots$ are not lost if we replace it by another countable system which contains it. We do this in such a way that the increased system $\overline{T}_1, \overline{T}_2, \cdots$ possesses these further properties:

(1) It contains $T' \cdot T''$ along with T', T''.

(2) It contains $a_0 T'$ along with T' for any $a_0 \in \mathfrak{G}$.

These being understood, we have

$$\mathbf{M} = \mathbf{R}(\bar{U}_{a_0}, \overline{L}_{\chi_{T_i}(x)}; a_0 \in \mathfrak{G}_1, i = 1, 2, \cdots). \tag{5.2.δ}$$

Consider now the finite groups $\mathfrak{G}_1, \mathfrak{G}_2, \cdots$ of (i). $\overline{T}_1, \cdots, \overline{T}_n$ generates, with the help of the operations (1) and (2) when a_0 in the latter is restricted to $\mathfrak{G}_n$, a finite subsystem of $\overline{T}_1, \overline{T}_2, \cdots$,—say the set $(T_i, i \in I_n)$. So I_n is a finite set. Clearly $I_1 \subseteq I_2 \subseteq \cdots$. I_n contains $1, 2, \cdots, n$. Hence $(1, 2, \cdots)$ is the set-theoretical sum of $I_1, I_2, \cdots$.

Consequently (5.2.δ) gives

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n, n = 1, 2, \cdots) \tag{5.2.ϵ}$$

with

$$\mathbf{N}_n = \mathbf{R}(\bar{U}_{a_0}, \overline{L}_{\chi_{T_i}(x)}; a_0 \in \mathfrak{G}_n, i \in I_n). \tag{5.2.ζ}$$

Thus $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots$ and therefore $\mathbf{M}$ is approximately finite by Def. 4.6.1 if the $\mathbf{N}_n$ are shown to be of finite order.[46]

Now the group character of $\mathfrak{G}_n$ and the construction of I_n show that the operators

$$\bar{U}_{a_0} \overline{L}_{\chi_{T_i}(x)}, \qquad a_0 \in \mathfrak{G}_n, \qquad i \in I_n \tag{5.2.η}$$

are reproduced by multiplication. As $\mathfrak{G}_n$, I_n are finite, these operators (5.2.η) form a finite set too. Hence they are a finite basis of $\mathbf{N}_n$ completing the proof.

This last result establishes the approximate finiteness of the examples (β), (γ) in [5], p. 208. Thus our Theorem XIV establishes their algebraic isomorphism and [6], p. 244, Theorem XI, their spatial isomorphism as announced by (v) in [5], p. 299. In order to include (α), p. 208, eod., we need this

LEMMA 5.2.3. *The condition (i) in Lemma 5.2.2 can be replaced by this condition*:

(ii) $\mathfrak{G}$ *is abelian.*

The proof of this lemma is somewhat complicated. It requires some rather deep results on the decompositions of mappings of measurable sets, which will be published elsewhere. We shall not pursue this matter further on this occasion.

§5.3 We proceed to expound the new (simplified) method for the construction of examples of factors in the case (II_1), announced in §5.1.

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[46] This time we really must use Def. 4.6.1 and it would not be possible to replace it by Def. 4.1.1—in contrast with footnote[44].

Let a group $\mathfrak{G}$ be given. For the moment $\mathfrak{G}$ could be finite or countably infinite, but we shall be forced to assume the latter at a subsequent stage (cf. Lemma 5.3.4 and the Remark after it).

Form a unitary vector space $\mathfrak{H}$ by using the elements of $\mathfrak{G}$ as indices for the vector components, i.e. $\mathfrak{H}$ is the set of all complex vectors $f = [x_a ; a \in \mathfrak{G}]$ with a finite $\sum_{a\in\mathfrak{G}} |x_a|^2$ and for $f = [x_a ; a \in \mathfrak{G}]$ and $g = [y_a ; a \in \mathfrak{G}]$, $(f, g) = \sum_{a\in\mathfrak{G}} x_a \bar{y}_a$.[47]

Now we introduce and discuss some operators in $\mathfrak{H}$.

LEMMA 5.3.1. *Define the operators*

$$\left.\begin{array}{ll} \text{(i)} & U_a[x_a ; a \in \mathfrak{G}] = [x_{aa_0} ; a \in \mathfrak{G}] \\ \text{(ii)} & V_{a_0}[x_a , a \in \mathfrak{G}] = [x_{a^{-1}a} ; a \in \mathfrak{G}] \end{array}\right\} \text{ for any } a_0 \in \mathfrak{G}.$$

$$\text{(iii)} \qquad W[x_a ; a \in \mathfrak{G}] = [x_{a^{-1}} ; a \in \mathfrak{G}]$$

These operators are unitary and

$$W = W^{-1}, \qquad WU_{a_0}W = V_{a_0}.^{48}$$

In other words, $f \to Wf$ is an involutory spatial automorphism of $\mathfrak{H}$ which interchanges U_{a_0} with V_{a_0}.

PROOF. All assertions are immediately verified.

We introduce, as usual, the complete normalized orthogonal system

$$\text{(5.3.}\alpha\text{)} \qquad \varphi_a , \qquad a \in \mathfrak{G},$$

in $\mathfrak{H}$ where $\varphi_a = [\delta_{a,b} ; b \in \mathfrak{G}]$. Thus $f = [x_a ; a \in \mathfrak{G}]$ is equivalent to $f = \sum_{a\in\mathfrak{G}} x_a\varphi_a$ and

$$\text{(5.3.}\beta\text{)} \qquad U_{a_0}\varphi_a = \varphi_{aa_0^{-1}}, \qquad V_{a_0}\varphi_a = \varphi_{a_0a}, \qquad W\varphi_a = \varphi_{a^{-1}}.$$

LEMMA 5.3.2. *Let* **I** *be the set of all U_{a_0} and* **J** *the set of all V_{a_0}. Form for each bounded operator A in $\mathfrak{H}$, its matrix $\{\alpha_{a,b}\}$ $(a, b \in \mathfrak{G})$ in the usual way.*[49]

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[47] This $\mathfrak{H}$ coincides with the $\mathfrak{H}_{\mathfrak{G}}$ of [5], p. 194, Def. 12.1.4. The only difference is that we now write x_a in place of the $f(a)$, loc. cit.

[48] One also verifies immediately

$$U_{a_1} \cdot U_{a_0} = U_{a_1a_0}, \qquad V_{a_1}V_{a_0} = V_{a_1a_0}$$

i.e. $a \to U_a$ and $a \to V_a$ are both representations of $\mathfrak{G}$. They correspond in fact to the regular representation of $\mathfrak{G}$.

[49] The definition is

$$\text{([49], }\alpha\text{)} \qquad \alpha_{a,b} = (A\varphi_a , \varphi_b) = (\varphi_a , A^*\varphi_b).$$

Hence

$$\|A\varphi_a\|^2 = \sum_{b\in\mathfrak{G}} |(A\varphi_a , \varphi_b)|^2 = \sum_{b\in\mathfrak{G}} |\alpha_{a,b}|^2$$

$$\|A^*\varphi_b\|^2 = \sum_{a\in\mathfrak{G}} |\varphi_a , A^*\varphi_b)|^2 = \sum_{a\in\mathfrak{G}} |\alpha_{a,b}|^2$$

i.e.

$$\text{([49], }\beta\text{)} \qquad \text{all } \sum_{b\in\mathfrak{G}} |\alpha_{a,b}|^2, \quad \sum_{a\in\mathfrak{G}} |\alpha_{a,b}|^2 \text{ are finite.}$$

(*continued*)

Then $A \in \mathbf{I}'$ if and only if $\alpha_{a,b}$ has the form $\alpha_{a,b} = \eta_{ab^{-1}}$ and $A \in \mathbf{J}'$ if and only if $\alpha_{a,b}$ has the form $\alpha_{a,b} = \eta_{a^{-1}b}$ i.e. $\alpha_{a,b}$ depends on the value of ab^{-1} only or on the value of $a^{-1}b$ only, respectively. (We do not determine for which systems η_c, $c \in \mathfrak{G}$ a bounded A to which the given η_c corresponds, actually does exist.)

PROOF. If $A \sim \{\alpha_{a,b}\}$, then (5.3.β) gives $U_{a_0}^{-1} A U_{a_0} \sim \{\alpha_{aa^{-1},ba_0^{-1}}\}$.[50] Now $A \in \mathbf{I}'$ means that A commutes with all U_{a_0} i.e. that

$$\alpha_{a,b} = \alpha_{aa^{-1},ba_0^{-1}} \quad \text{for all} \quad a_0 \in \mathfrak{G}. \tag{5.3.γ}$$

Write $\eta_c = a_{c,1}$. Then (5.3.γ) with $a_0 = b$ gives

$$\alpha_{a,b} = \eta_{ab^{-1}}. \tag{5.3.δ}$$

Conversely (5.3.δ) implies (5.3.γ). Thus (5.3.δ) is characteristic for $A \in \mathbf{I}'$. This proves our statement concerning $\mathbf{I}'$.

If $A \sim \{\alpha_{a,b}\}$ then $WAW \sim \{\alpha_{a^{-1},b^{-1}}\}$.[50] Hence the automorphism $f \to Wf$ of $\mathfrak{H}$ carries (5.3.δ) into

$$\alpha_{a,b} = \eta_{a^{-1}b}. \tag{5.3.ϵ}$$

Since it carries $\mathbf{I}$ into $\mathbf{J}$ it takes $\mathbf{I}'$ into $\mathbf{J}'$. Therefore (5.3.ϵ) is characteristic for $A \in \mathbf{J}'$. This proves our statement concerning $\mathbf{J}'$.

LEMMA 5.3.3. $\mathbf{R}(\mathbf{I}) = \mathbf{J}'$, $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$.

PROOF. Clearly V_{a_0} commutes with all U_{a_0}. So $V_{a_0} \in \mathbf{I}'$ and hence $\mathbf{J} \subseteq \mathbf{I}'$. As $\mathbf{I}'$ is a ring, this implies $\mathbf{R}(\mathbf{J}) \subseteq \mathbf{I}'$.

If $A \in \mathbf{I}'$, $B \in \mathbf{J}'$ then $A \sim \{\eta_{ab^{-1}}\}$, $B \sim \{\theta_{a^{-1}b}\}$. Direct computation shows that $AB = BA$ (use the matrix computation rules as mentioned at the end of footnote [49]). Thus $A \in \mathbf{J}''$ and hence $\mathbf{I}' \subseteq \mathbf{J}''$. Since $1 \in \mathbf{J}$, $\mathbf{J}'' = \mathbf{R}(\mathbf{J})$ (cf. [2], p. 397). Therefore $\mathbf{I}' \subseteq \mathbf{R}(\mathbf{J})$.

Finally

$$\begin{aligned}(Af, \varphi_b) &= (f, A^*\varphi_b) = \textstyle\sum_{a\in\mathfrak{G}} (f, \varphi_a)(\varphi_a, A^*\varphi_b) \\ &= \textstyle\sum_{a\in\mathfrak{G}} \alpha_{a,b}(f, \varphi_a)\end{aligned}$$

i.e.

$$\begin{cases}\text{If} \quad f = \sum_{a\in\mathfrak{G}} x_a \varphi_a, \qquad Af = \sum_{a\in\mathfrak{G}} y_a \varphi_a \\ \text{then} \quad y_b = \sum_{a\in\mathfrak{G}} \alpha_{a,b} x_a.\end{cases} \tag{^{49}, γ}$$

The matrix operations for these (complex numerical) matrices are defined in the same way as for the (operator) matrices of B_p) in §2.4.

[50] Use ([49], α) and (5.3. β). Then

$$\begin{aligned}(U_{a_0}^{-1} A U_{a_0}\varphi_a, \varphi_b) &= (AU_{a_0}\varphi_a, U_{a_0}\varphi_b) \\ &= (A\varphi_{aa_0^{-1}}, \varphi_{ba_0^{-1}})\end{aligned}$$

$$(WAW\varphi_a, \varphi_b) = (AW\varphi_a, W\varphi_b) = (A\varphi_{a^{-1}}, \varphi_{b^{-1}})$$

Hence $A \backsim \{\alpha_{a,b}\}$ implies $U_{a_0}^{-1} A U_{a_0} \backsim \{\alpha_{aa_0^{-1},ba_0^{-1}}\}$ and $WAW \backsim \{\alpha_{a^{-1},b^{-1}}\}$.

The results of the two preceding paragraphs imply $\mathbf{R}(\mathbf{J}) = \mathbf{I}'$. The spatial automorphism $f \to Wf$ of $\mathfrak{H}$ interchanges $\mathbf{I}$ with $\mathbf{J}$ and thus we obtain $\mathbf{R}(\mathbf{I}) = \mathbf{J}'$ too. The proof is now complete.

LEMMA 5.3.4. *Put* $\mathbf{M} = \mathbf{R}(\mathbf{I}) = \mathbf{J}'$. *Then* $\mathbf{M}' = \mathbf{R}(\mathbf{J})$. $\mathbf{I}'$ *is a factor if and only if* $\mathfrak{G}$ *fulfills this condition*:

(i) *For every* $a \in \mathfrak{G}$, $a \neq 1$, *the class of* a, $\mathfrak{L}_a = (c^{-1}ac;\ c \in \mathfrak{G})$ *is infinite.*

PROOF. Clearly $\mathbf{M} = (\mathbf{R}(\mathbf{I}))' = \mathbf{I}'$.

Now consider an $A \in \mathbf{M} \cdot \mathbf{M}'$. Put $A \sim \{\alpha_{a,b}\}$. $A \in \mathbf{M} \cdot \mathbf{M}'$ implies that both

$$\alpha_{a,b} = \eta_{a^{-1}b} \quad \text{and} \quad \alpha_{a,b} = \theta_{ab^{-1}} \tag{5.3.ζ}$$

hold. Put $b = 1$. This gives $\eta_{a^{-1}} = \theta_a$. Thus (5.3.ζ) is equivalent to

$$\alpha_{a,b} = \eta_{a^{-1}b} = \eta_{ba^{-1}}. \tag{5.3.η}$$

The second equation in (5.3.η) may be rewritten by replacing a, b by b, ab. Then it becomes $\eta_{b^{-1}ab} = \eta_a$ i.e. it states that η_d is a constant for all $d \in \mathfrak{L}_a$. So (5.3.η) is equivalent to this

$$\alpha_{a,b} = \eta_{a^{-1}b},\ \eta_d \text{ is constant for all } d \in \mathfrak{L}_a. \tag{5.3.θ}$$

We now prove that (i) is characteristic.

Sufficiency: Assume that (i) is true. If $a \neq 1$ then $\mathfrak{L}_a$ is infinite. Hence the finiteness of $\sum_{a \in \mathfrak{G}} |\ \eta_d\ |^2$ [51] together with (5.3.θ) imply $\eta_a = 0$ if $a \neq 1$.

Thus $$\alpha_{a,b} = \eta_{a^{-1}b} \begin{cases} = \eta_1 & \text{for } a = b \\ = 0 & \text{otherwise.} \end{cases}$$

or $A = \eta \cdot 1$ and so $\mathbf{M}$ is a factor.

Necessity: Assume that (i) is not true. Choose an $a_0 \neq 1$ with $\mathfrak{L}_{a_0}$ finite. Define

$$\alpha_{a,b} = \eta_{a^{-1}b},\ \eta_d \begin{cases} = 1 & \text{for } a \in \mathfrak{L}_a \\ = 0 & \text{otherwise.} \end{cases} \tag{5.3.ι}$$

One verifies that the $A \sim \{\alpha_{a,b}\}$ may be expressed as $A = \sum_{d \in \mathfrak{L}_a} U_d$. [52] Thus $A \in \mathbf{M} \cdot \mathbf{M}'$ by (5.3.θ) and (5.3.ι).

If $\mathbf{M}$ were a factor, this would necessitate $A = \alpha \cdot 1$ or $A \sim \{\alpha\delta_{a,b}\}$ where $\delta_{a,b} = 1$ for $a = b$ and $\delta = 0$ otherwise. Consequently

$$\alpha_{a,b} = \alpha\delta_{a,b}. \tag{5.3.κ}$$

Now (5.3.ι) and (5.3.κ) give for $a = 1$ and $b = a_0$ (remember $a_0 \neq 1$) $\alpha_{1,a_0} = 1$ and $\alpha_{1,a_0} = 0$ respectively. This contradiction shows that $\mathbf{M}$ is not a factor.

[51] The finiteness of $\sum_{d \in \mathfrak{G}} |\ \eta_d\ |^2$ follows immediately from the first formula of $(^{49}, \beta)$ with $a = 1$:

$$\sum_{b \in \mathfrak{G}} |\ \alpha_{1,b}\ |^2 = \sum_{b \in \mathfrak{G}} |\ \eta_b\ |^2.$$

[52] Observe that the sum can be formed because $\mathfrak{L}_a$ is finite.

This last result forces us to assume that $\mathfrak{G}$ is infinite, since even its classes $\mathfrak{L}_a$, $a \neq 1$ are to be infinite. Groups $\mathfrak{G}$ with this property are easy to construct. We shall give some examples at the end of §5.5 and in §5.6. Clearly no such group can be abelian.

There is, however, another circumstance worth pointing out before we go further. Our construction is obviously nothing but an extension of Frobenius's group numbers to infinite groups.[53] The unitary space $\mathfrak{H}$ and its operators are only technical devices to take care of convergence difficulties. Both the U_a and V_a furnish representations of $\mathfrak{G}$ (cf. footnote [48]), and so both $\mathbf{M} = \mathbf{R}(\mathbf{I})$ and $\mathbf{M}' = \mathbf{R}(\mathbf{J})$ are the equivalents of Frobenius's group numbers. But in the case of finite $\mathfrak{G}$ this procedure does not give a factor. The group numbers notoriously possess other central elements than the $\alpha \cdot 1$. There are as many linearly independent ones as there are different classes $\mathfrak{L}_a$ in $\mathfrak{G}$. The real meaning of our Lemma 5.3.4 appears to be this: The role of all classes in $\mathfrak{G}$, when $\mathfrak{G}$ is finite, is now taken over by the finite classes.[54] For finite $\mathfrak{G}$ this distinction is vacuous, but for the infinite $\mathfrak{G}$ it is now seen to be decisive.

In other words: We have essentially merely extended the construction of Frobenius group numbers to infinite $\mathfrak{G}$. For finite $\mathfrak{G}$ this can never give a factor; indeed, the usefulness of the group numbers lies in that case in the opposite direction. For infinite $\mathfrak{G}$ however, our treatment of the convergence questions makes this construction perfectly adequate to produce factors.

We now determine the case to which these factors $\mathbf{M}$, $\mathbf{M}'$ belong, and a few other characteristics.

The condition (i) in Lemma 5.3.4 is assumed to be fulfilled.

LEMMA 5.3.5. (i) $\mathbf{M}$, $\mathbf{M}'$ *are coupled factors both in case* (II_1).

(ii) *In their standard normalization, the* C *of* [5], *p.* 182, *Theorem X, is* 1.

(iii) *If* $A \in \mathbf{M}$, $(A \in \mathbf{M}')$ *then* $Tr_{\mathbf{M}}(A)$, $(Tr_{\mathbf{M}'}(A))$ *has the value* η_1 *for the* η_c *of Lemma* 5.3.2.

PROOF. We proceed in a somewhat changed order.

Define for $A \in \mathbf{M}$, a quantity $T'(A) = \eta_1$ for the η_c of Lemma 5.3.2. Let us check for this $T'(A)$ the properties (i)–(vi′) in [6], p. 218.

(i)–(iii), (v) loc. cit., are obvious.

If $A \sim \{\alpha_{a,b}\} = \{\eta_{a^{-1}b}\}$, $B \sim \{\beta_{a,b}\} = \{\theta_{a^{-1}b}\}$ then clearly $AB \sim \{\gamma_{a,b}\} = \{\zeta_{a^{-1}b}\}$ where $\zeta_a = \sum_{c \in \mathfrak{G}} \eta_c \theta_{ac^{-1}}$. (Use the matrix computation rules indicated at the end of footnote [49].) Hence $T'(AB) = \zeta_1 = \sum_{c \in \mathfrak{G}} \eta_c \theta_{c^{-1}}$. This expression is symmetric in η_a and θ_a. Hence

$$T'(AB) = T'(BA) = \sum_{c \in \mathfrak{G}} \eta_c \theta_{c^{-1}}. \tag{5.3.μ}$$

This proves (vi), loc. cit., and (vi′) eod., by replacing A, B by $U^{-1}A$, U.

[53] We treat these groups as discrete groups and use everywhere sums not integrals. The familiar generalization of Peter-Weyl to continuous groups and of von Neumann to almost-periodic groups where integrals and means are used, do not show the peculiarities of our present discussion. In those cases the infinite group behaves almost entirely like the finite group.

[54] Concerning the connection between factors and representation theory, cf. also [5], pp. 118–120, Introduction, §3.

As $A^* = \{\overline{\alpha_{b,a}}\} = \{\xi_{a^{-1}b}\}$ with $\xi_a = \overline{\eta_{a^{-1}}}$, (5.3.$\mu$) yields for $B = A^*$

$$T'(A^*A) = \sum_{c\in\mathfrak{G}} |\eta_c|^2. \tag{5.3.ν}$$

This proves (iv), (iv'), loc. cit.

Thus all desired relations are established.

If we consider $T'(E)$ only for projections $E \in \mathbf{M}$ then the above relations allow us to assert the validity of (ii), (iii) in [5], p. 165, Def. 8.2.1. Also $T'(1) = 1 \neq 0, \infty$. Hence, by [5], p. 170, Lemma 8.3.5, this $T'(E)$ is a relative dimension function of $\mathbf{M}$ and $\mathbf{M}$ is in a finite case.

The cases (I_n), $n = 1, 2, \cdots$ are excluded, since $\mathbf{M}$ is not a finite ring. Hence $\mathbf{M}$ is in the case (II_1).

Now we conclude by [6], p. 218, Property IV, that $Tr_{\mathbf{M}}(A) = T'(A)$ i.e. that $Tr_{\mathbf{M}}(A) = \eta_1$.

Thus our (i), (iii) hold for $\mathbf{M}$. The spatial automorphism $f \to Wf$ in $\mathfrak{H}$ shows that they hold for $\mathbf{M}'$ in view of Lemma 5.3.1.

Let us now consider our (ii). Observe that by (5.3.β) $\mathfrak{M}_{\varphi_1}^{\mathbf{M}}$ as well as $\mathfrak{M}_{\varphi_1}^{\mathbf{M}'}$ (cf. [5], p. 143, Def. 5.1.1) contain all φ_a, $a \in \mathfrak{G}$. Hence both are $\mathfrak{H}$. It follows that in the standard normalization.

$$D_{\mathbf{M}}(\mathfrak{M}_{\varphi_1}^{\mathbf{M}'}) = D_{\mathbf{M}'}(\mathfrak{M}_{\varphi_1}^{\mathbf{M}}) = 1.$$

Therefore [5], p. 182, Theorem X, gives $c = 1$ as asserted.

LEMMA 5.3.6. *Consider an $A \in \mathbf{M}$ ($A \in \mathbf{M}'$) and its η_c by Lemma 5.3.2. Then we have*

(i) $$[[A]] = (\sum_{c\in\mathfrak{G}} |\eta_c|^2)^{\frac{1}{2}}$$

(ii) $$A = \sum_{a\in\mathfrak{G}} \eta_c U_c, \qquad (A = \sum_{a\in\mathfrak{G}} \eta_c V_c)$$

in the sense of metric convergence in $\mathbf{M}$, ($\mathbf{M}'$) irrespective of the order in which the $c \in \mathfrak{G}$ are gone through.

PROOF. The spatial automorphism $f \to Wf$ of $\mathfrak{H}$ interchanges $\mathbf{M}$ and $\mathbf{M}'$. Therefore it suffices to consider the $A \in \mathbf{M}$.

Ad (i): By Equation (5.3.ν) in the proof of Lemma 5.3.5,

$$[[A]]^2 = Tr_{\mathbf{M}}(A^*A) = T'(A^*A) = \sum_{c\in\mathfrak{G}} |\eta_c|^2.$$

Hence $[[A]] = (\sum_{c\in\mathfrak{G}} |\eta_c|^2)^{\frac{1}{2}}$.

Ad (ii): Choose any enumeration $a^{(1)}, a^{(2)}, \cdots$ of $\mathfrak{G}$. One verifies immediately that for $A^{(n)} = \sum_{k=1}^{n} \eta_{a^{(k)}} U_{a^{(k)}}$

$$A^{(n)} \sim \{\eta^{(n)}_{a^{-1}b}\} \quad \text{with} \quad \eta_c^{(n)} \begin{cases} = \eta_c & \text{for } c = a^{(1)}, \cdots, a^{(n)} \\ = 0 & \text{otherwise.} \end{cases}$$

Hence

$$A - A^{(n)} \sim \{\eta^{(n)1}_{a^{-1}b}\} \quad \text{with} \quad \eta_c^{(n)1} \begin{cases} = 0 & \text{for } c = a^{(1)}, \cdots, a^{(n)} \\ = \eta_c & \text{otherwise.} \end{cases}$$

Consequently (i) above gives

$$[[A - A^{(n)}]] = (\sum_{c\in\mathfrak{G}} | \eta_c^{(n)1} |^2)^{\frac{1}{2}} = (\sum_{c\in\mathfrak{G},\, c\neq a^{(1)},\cdots,a^{(n)}} | \eta_c |^2)^{\frac{1}{2}}$$
$$= (\sum_{i=n+1}^{\infty} | \eta_{a^{(i)}} |^2)^{\frac{1}{2}}$$

This is equivalent to

$$[[A - \sum_{k=1}^{n} \eta_{a^{(k)}} U_{a^{(k)}}]] = (\sum_{k=n+1}^{\infty} | \eta_{a^{(k)}} |^2)^{\frac{1}{2}}. \tag{5.3.o}$$

Now $\sum_{k=1}^{\infty} | \eta_{a^{(k)}} |^2 = \sum_{c\in\mathfrak{G}} | \eta_c |^2$ is finite by (i) above. Hence the right hand side of (5.3.o) tends to 0 as $n \to \infty$. This means that $\sum_{k=1}^{\infty} \eta_{a^{(k)}} U_{a^{(k)}}$ converges metrically to A.

Since this holds for any enumeration $a^{(1)}, a^{(2)}, \cdots$ of $\mathfrak{G}$, our assertion is established.

§5.4 As we pointed out, the construction of §5.3 is closely related to that of [5], pp. 192–209, Part IV. The group $\mathfrak{G}$ plays the same role in both cases, and the difference consists in the disappearance of the space S.

This space played an important part in the construction referred to above: Every $a \in \mathfrak{G}$ had to induce a mapping $x \to xa$ in S (cf. loc. cit., p. 195, Def. 12.1.5). The absence of S from our present construction may also be interpreted by saying that S is now a one-element set: $S = (x_0)$ and that for every $a \in \mathfrak{G}$, $x_0 a = x_0$.

Now the difference between our two constructions can be expressed as follows: In [5], $\mathfrak{G}$ was entirely unrestricted, but S had to obey (among other things) (iii) in Def. 12.1.5, loc. cit., p. 195: for $a \neq 1$ and $x \in S$ always $xa \neq x$.[55] In §5.3, $\mathfrak{G}$ had to fulfill (i) of Lemma 5.3.4, but the (iii) referred to was most flagrantly violated.[56, 57] Thus our two procedures correspond to two different ways of securing the factor character of **M** in what is fundamentally the same construction. In one case the whole burden of the necessary restrictions was thrown on S in the other case, on $\mathfrak{G}$.

It would be possible to use a more general arrangement of which both these procedures are special cases. The necessary restriction would then be of a mixed type, affecting the structure of S and $\mathfrak{G}$.

We shall not consider this question in more detail at this time.

§5.5 A second connection between our two constructions which deserves some attention can be obtained as follows.

Let an arbitrary countably infinite group $\mathfrak{G}$ be given. We shall try to use it

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[55] Except for an x set of measure zero.

[56] We have $x_0 a = x_0$ and $\mu((x_0)) = \mu(S) \neq 0$.

[57] There is no difference between the two constructions as regards the other conditions of Def. 12.1.5, loc. cit., p. 195. In particular, $\mathfrak{G}$ is ergodic in our present construction, since $S = (x_0)$ is a one-element set.

for the construction of [5] referred to above, that is to find an S which fulfills all preliminary conditions of that construction in conjunction with the given $\mathfrak{G}$.

This construction runs as follows:[58]

(I) Let S be the set of all systems $x = [\alpha_a ; a \in \mathfrak{G}]$ where each $\alpha_a = 0, 1$. Let $\mathfrak{S}$ be the set of those $x = [\alpha_a ; a \in \mathfrak{G}]$ for which $\alpha_a = 0$ except for a finite number of a's.

Define in S: If $x = [\alpha_a ; a \in \mathfrak{G}]$, $y = [\beta_a ; a \in \mathfrak{G}]$ then $x \oplus y = [\gamma_a ; a \in \mathfrak{G}]$ where $\gamma_a = \alpha_a + \beta_a (\mathrm{mod}\ 2)$, $\gamma_a = 0$ or 1. S is clearly a (commutative) group with the "unit" $0 = [0; a \in \mathfrak{G}]$ and $\mathfrak{S}$ is an enumerably infinite subgroup of S.

Define further in S: If $x = [\alpha_a ; a \in \mathfrak{G}]$ and $a_0 \in \mathfrak{G}$ then $xa_0 = [\alpha_{a_0 a} ; a \in \mathfrak{G}]$. Then the mappings $x \to xa_0$ represent the group $\mathfrak{G}$ by permutations of S, all of which carry $\mathfrak{S}$ into itself.

Now choose an arbitrary but fixed enumeration $a^{(1)}, a^{(2)}, \cdots$ of $\mathfrak{G}$. For S we use the mapping

$$\Xi \qquad x = [\alpha_a; a \in \mathfrak{G}] \to \xi(x) = \sum_{m=1}^{\infty} \frac{\alpha_{a^{(m)}}}{2^m}$$

of S on the numerical interval $0 \leqq \xi \leqq 1$. Except for the Ξ image of $\mathfrak{S}$ the set of all dyadically rational numbers, which is a set of Lebesgue measure 0 this mapping is one-to-one. So the common (exterior) Lebesgue measure in $0 \leqq \xi \leqq 1$ is mapped by the inverse of Ξ on a Lebesgue measure in S with all its usual properties.

We shall now consider $\mathfrak{G}$ as the "group" and S (with the "mappings" $x \to xa_0$ for $x \in S$, $a_0 \in \mathfrak{G}$) as the "space" described in [5], pp. 192–195. In the sense of [5], p. 195, Def. 12.1.5, $\mathfrak{G}$ is an m-group and ergodic in S.

First we show the m-group character. Ad (i), loc. cit. This is concerned with the preservation of outer measure of the measure determining sets $\{T_i\}$. In view of the definition of measure in S given in the previous paragraph, we may take for a set of T_i's the set of images, under the inverse of Ξ, of ξ-intervals $k/2^m \leqq \xi < (k+1)/2^m$. An image of a ξ-interval $k/2^m \leqq \xi < (k+1)/2^m$ is given by specifying the values of $\alpha_{a^{(1)}}, \cdots, \alpha_{a^{(m)}}$. Hence it is mapped by $x \to xa_0$ onto a set, in which the values of $\alpha_{a_0^{-1}a^{(1)}}, \cdots, \alpha_{a_0^{-1}a^{(m)}}$ are specified. Choose n so that $a_0^{-1}a^{(1)}, \cdots, a_0^{-1}a^{(m)}$ occur among the $a^{(1)}, \cdots, a^{(n)}$. Then the above set is the sum of 2^{n-m} sets, each of which is defined by specifying the values of $\alpha_{a^{(1)}}, \cdots, \alpha_{a^{(n)}}$, i.e. each of which corresponds to an interval $l/2^n \leqq \xi < (l+1)/2^n$. Thus the set corresponding to the ξ interval $k/2^m \leqq \xi < (k+1)/2^m$ having measure $1/2^m$ is mapped on a set of measure $2^{n-m}/2^n = 2^{-m}$. Thus the measure of each such is conserved. Ad (ii) which states that $(xa_0)b_0 = x(a_0 b_0)$ can be verified by a direct computation.[59] Ad (iii), states that if $a_0 \neq 1$, $xa_0 = x$ holds

[58] The construction (I)–(V) which follows is in many respects analogous to the construction (I)–(VII) in [7], pp. 72–77. The main differences are these: We use $\mathfrak{G}$ and not $\mathfrak{S}$ as the group of mappings in S, and the latter parts of the two constructions pursue different aims.

[59] Let $x = [\alpha_a , a \in \mathfrak{G}]$, $xa_0 = [\beta_a , a \in \mathfrak{G}]$, $(xa_0)b_0 = [\gamma_a , a \in \mathfrak{G}]$. Then $\beta_a = \alpha_{a_0 a}$ and $\gamma_a = \beta_{b_0 a} = \alpha_{a_0 (b_0 a)} = \alpha_{(a_0 b_0) a}$. Hence $(xa_0)b_0 = [\alpha_{(a_0 b_0)a} , a \in \mathfrak{G}] = x(a_0 b_0)$.

only for a set of measure zero. For $x = [\alpha_a ; a \in \mathfrak{G}]$, $xa_0 = x$ is equivalent to $\alpha_{a_0 a} = \alpha_a$ for all a. But since $a_0 \in \mathfrak{G}$, $a_0 \neq 1$ we can construct an infinite sequence $a_1, a_2, \cdots$ such that the $a_1, a_2, \cdots, a_0a_1, a_0a_2, \cdots$ are all different.[60] For any m, let $n = n(m)$ be such that $a^{(1)}, \cdots, a^{(n)}$ includes $a_1, \cdots, a_m, a_0a_1, \cdots, a_0a_m$. Then the x-set $S^{(m)}$ for which $\alpha_{a_0 a_i} = \alpha_{a_i}$ for $i = 1, \cdots, m$ consists of 2^{n-m} sets $I^{(n)}$ in each of which the $\alpha_{a^{(1)}, \cdots, \alpha_{a^{(n)}}}$ have specified values. Such a set $I^{(n)}$ is the image of a ξ interval of measure 2^{-n} and hence $S^{(m)}$ has measure $2^{n-m} \cdot 2^{-n} = 2^{-m}$. Now the x-set for which $\alpha_{a_0 a} = \alpha_a$ for all $a \in \mathfrak{G}$ is in $S^{(m)}$ for every m. Thus it has measure zero. Since this is also the set for which $xa_0 = x$ we have shown, Ad (iii).

The ergodicity will be established in (III) below.

(II) Form for these S, $\mathfrak{G}$ the spaces $\mathfrak{H}_S$ and $\mathfrak{H}_{\mathfrak{G}S}$ of all complex-valued functions $f(x)$ resp. $F(x, a)$, $x \in S$, $a \in \mathfrak{G}$ which are measurable in x and with a finite $\int_S |f(x)|^2 dx$ resp. $\sum_{a \in \mathfrak{G}} \int_S |F(x, a)|^2 dx$. Thus $(f, g) = \int_S f(x)\overline{g(x)}\, dx$, $(F, G) = \sum_{a \in \mathfrak{G}} \int_S F(x, a)\overline{G(x, a)}\, dx$ (cf. [5], p. 194). Form the bounded operators

$$\left.\begin{aligned} U_{a_0} f(x) &= f(xa_0) \\ U_{a_0} F(x, a) &= F(xa_0, aa_0) \end{aligned}\right\} a_0 \in \mathfrak{G}.$$

$$L_{\varphi(x)} F(x, a) = \varphi(x) F(x, a) \quad (\varphi(x) \text{ bounded and measurable})$$

(loc. cit., pp. 196, 198–199) and the ring $\mathbf{M}$ in $\mathfrak{H}_{\mathfrak{G}S}$ which the U_{a_0}, $L_{\varphi(x)}$ generate (loc. cit., p. 200):

(5.5.α) $\qquad \mathbf{M} = \mathbf{R}(U_{a_0}, L_{\varphi(x)}, a_0 \in \mathfrak{G}, \varphi(x) \text{ bounded and measurable}).$

Before we go on we form the following system of functions in $\mathfrak{H}_{\mathfrak{G}S}$

$$w_{\bar{x}}(x) = (-1)^{\sum_{a \in \mathfrak{G}} \bar{\alpha}_a \alpha_a} \qquad (\bar{x} = [\bar{\alpha}_a, a \in \mathfrak{G}] \in \mathfrak{S}, x = [\alpha_a, a \in \mathfrak{G}] \in S.)^{62}$$

It is easy to verify that these functions (for all $\bar{x} \in \mathfrak{S}$) form a complete normalized orthogonal system in $\mathfrak{H}_S$.[63] Consequently the functions

$$F_{\bar{x}, \bar{a}}(x, a) = \omega_{\bar{x}}(x) \delta_{a, \bar{a}}, \qquad (x \in \mathfrak{S}, \bar{a} \in \mathfrak{G})$$

form a complete normalized orthogonal system in $\mathfrak{H}_{\mathfrak{G}, S}$.

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[60] If the $a_0, a_0^2, \cdots$ are all distinct, let $a_i = a_0^{2i-1}$. The result readily follows in this case. Otherwise, a_0 is of finite order, $p = 2, 3, \cdots$ ($p \neq 1$ since $a_0 \neq 1$). For each a we can define S_a as the set of $b \in \mathfrak{G}$ for which $b = a_0^k a$ for some $k = 0, 1, \cdots, p - 1$. It is easily seen how we can choose a sequence $a_1, a_2, \cdots$ such that the S_{a_i} are mutually exclusive. For this sequence we have that the $a_1, a_2, \cdots, a_0a_1, a_0a_2, \cdots$ are all different. This readily follows from the facts that the S_{a_i} are mutually exclusive and $a_0 \neq 1$.

[62] Owing to $\bar{x} = [\bar{\alpha}_a, a \in \mathfrak{G}] \in \mathfrak{S}$ we have $\bar{\alpha}_a \neq 0$ only for a finite set of a's. Hence the sum $\sum_{a \in \mathfrak{G}} \bar{\alpha}_a \alpha_a$ is really finite.

[63] In this particular case the group nature of $\mathfrak{G}$ is irrelevant. Hence we can replace $\mathfrak{G}$ by any other countably infinite set. We replace the $a \in \mathfrak{G}$ by $a = 1, 2, \cdots$. After this substitution our $\omega_{\bar{x}}(x)$ become the well-known complete normalized orthogonal set of Walsh-Rademacher functions. It was also considered in [7], p. 74, under the name of $\omega_a(x)$.

A familiar argument shows that the $\varphi(x)$ in (5.5.α) can be restricted considerably without changing the ring $\mathbf{M}$. Indeed:

First it suffices, for reasons of continuity, to consider those $\varphi(x)$ only which are finite linear aggregates of the $\omega_{\bar{x}}(x)$.[64] Second, we may now restrict the $\varphi(x)$ to the $\omega_{\bar{x}}(x)$ themselves. So we have

$$(5.5.\beta) \qquad \mathbf{M} = \mathbf{R}(\bar{U}_{a_0}, \overline{L_{\omega_{\bar{x}}(x)}}; a_0 \in \mathfrak{G}, \bar{x} \in \mathfrak{S}).$$

And as 1 occurs among the $\overline{U}_{a_0}$ (for $a_0 = 1$) and among the $\overline{L_{\omega_{\bar{x}}(x)}}$ (for $\bar{x} = 0$), we can equally write:

$$(5.5.\gamma) \qquad \mathbf{M} = \mathbf{R}(\bar{U}_{a_0}\overline{L_{\omega_{\bar{x}}(x)}}; a_0 \in \mathfrak{G}, \bar{x} \in \mathfrak{S}).$$

(III) Clearly

$$U_{a_0}\omega_{\bar{x}}(x) = \omega_{\bar{x}}(xa_0) = \omega_{\bar{x}a_0^{-1}}(x).$$

Now consider any $f \in \mathfrak{H}_S$ with $U_{a_0}f = f$ for all $a_0 \in \mathfrak{G}$. Use the orthogonal expansion $f(x) = \sum_{\bar{x}\in\mathfrak{S}} \lambda_{\bar{x}}\omega_{\bar{x}}(x)$. Then the above formula gives $f = \sum_{\bar{x}\in\mathfrak{S}} \lambda_{\bar{x}}\omega_{\bar{x}}$, $U_{a_0}f = \sum_{\bar{x}\in\mathfrak{S}} \lambda_{\bar{x}}\omega_{\bar{x}a_0^{-1}} = \sum_{\bar{x}\in\mathfrak{S}} \lambda_{\bar{x}a_0}\omega_{\bar{x}}$. Hence $\lambda_{\bar{x}} = \lambda_{\bar{x}a_0}$. As $||f||^2 = \sum_{\bar{x}\in\mathfrak{S}} |\lambda_{\bar{x}}|^2$ is finite, this implies that $\lambda_{\bar{x}} = 0$ if there are infinitely many distinct $\bar{x}a_0$, $a_0 \in \mathfrak{G}$. Now this is the case for all $\bar{x} \in \mathfrak{S}$, $\bar{x} \neq 0$.[65] So $\bar{x} \neq 0$ implies $\lambda_{\bar{x}} = 0$. Thus $f(x) = \lambda_0\omega_0(x) = \lambda_0$.

So $U_{a_0}f = f$ (for all $a_0 \in \mathfrak{G}$) implies that f is a constant. Hence $\mathfrak{G}$ is ergodic.

(IV) Clearly

$$\omega_{\bar{x}}(x)\omega_{\bar{y}}(x) = \omega_{\bar{x}\oplus\bar{y}}(x)$$

$$\overline{L_{\omega_{\bar{x}}(x)}}F_{\bar{y},\bar{b}}(x, a) = \omega_{\bar{x}}(x)F_{\bar{y},\bar{b}}(x, a) = F_{\bar{x}\oplus\bar{y},\bar{b}}(x, a).$$

Next

$$\bar{U}_{\bar{a}}F_{\bar{y},\bar{b}}(x, a) = F_{\bar{y},\bar{b}}(x\bar{a}, a\bar{a}) = F_{\bar{y}\bar{a}^{-1},\bar{b}\bar{a}^{-1}}(x, a).$$

Combining these gives

$$(5.5.\epsilon) \qquad \bar{U}_{\bar{a}}\overline{L}_{\omega_{\bar{x}}(x)}F_{\bar{y},\bar{b}}(x, a) = F_{(\bar{x}\oplus\bar{y})\bar{a}^{-1},\bar{b}\bar{a}^{-1}}(x, a).$$

Now treat the index $\bar{y}$, $\bar{b}$ of $F_{\bar{y},\bar{b}}$ as one pair. Make these pairs into a group $\{\mathfrak{S}, \mathfrak{G}\}$ by defining

$$(5.5.\zeta) \qquad \{\bar{y}, \bar{b}\}\{\bar{z}, \bar{c}\} = \{\bar{y}\bar{c} \oplus \bar{z}, \bar{b}\bar{c}\}.$$

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[64] Cf. the argument of [7], pp. 73–74 (II).

[65] We prove this as follows: Put $\bar{x} = [\bar{\alpha}_a, a \in \mathfrak{G}]$. Since $\bar{x} \neq 0$ there is an a^* such that $\bar{\alpha}_{a^*} \neq 0$. Now as in footnote[60], we can construct an infinite sequence of the a's $\in \mathfrak{G}$ such that a_1a^*, a_2a^* are all different. Then $\bar{x}a_i^{-1} = [\bar{\alpha}_{a_ia^{-1}a}, a \in \mathfrak{G}]$ has for its a_ia^* component $\bar{\alpha}_{a^*} = 1$. Thus if we consider the $\bar{x}a_1^{-1}$, $\bar{x}a_2^{-1}$, $\cdots$ we find that there is an infinite number of a's $\in \mathfrak{G}$ such that the a component is not zero for at least one of the $\bar{x}a_i^{-1}$. Since each $\bar{x}a_i^{-1}$ can have at most a finite number of components different from zero, there must be an infinite number of distinct $\bar{x}a_i^{-1}$.

As $\bar{x} \oplus \bar{x} = 0$ (5.5.ζ) implies that

$$\{\bar{y}, \bar{b}\}\{\bar{x}, \bar{a}\}^{-1} = \{(\bar{x} \oplus \bar{y})\bar{a}^{-1}, \bar{b}\bar{a}^{-1}\}.$$

Hence (5.5.ϵ) becomes

$$U_{\bar{a}} L_{\omega_{\mathfrak{F}}(x)} F_{\{\bar{y},\bar{b}\}}(x, a) = F_{\{\bar{y},\bar{b}\}\{\bar{x},\bar{a}\}^{-1}}(x, a). \tag{5.5η}$$

(V) Apply the construction of §5.3 to the group $\{\mathfrak{S}, \mathfrak{G}\}$ of (5.5.ζ) above, instead of its $\mathfrak{G}$. This is possible since $\{\mathfrak{S}, \mathfrak{G}\}$ fulfills (i) in Lemma 5.3.4; i.e. if $\{\bar{x}, \bar{a}\} \neq \{0, 1\}$ then there exist infinitely many different $\{\bar{y}, \bar{b}\}^{-1}\{\bar{x}, \bar{a}\}\{\bar{y}, \bar{b}\}$. Indeed, if $\bar{x} \neq 0$ then form $\{0, \bar{b}\}^{-1}\{\bar{x}, \bar{a}\}\{0, \bar{b}\} = \{\bar{x}\bar{b}, \bar{b}^{-1}\bar{a}\bar{b}\}$. There are infinitely many distinct $\bar{x}\bar{b}$ for $\bar{b} \in \mathfrak{G}$. Cf. footnote [65]. Thus our statement is true for $\bar{x} \neq 0$. If however $\bar{x} = 0$, then necessarily $\bar{a} \neq 1$. Form $\{\bar{y}, 1\}\{0, \bar{a}\}\{\bar{y}, 1\} = \{\bar{y}\bar{a} \oplus \bar{y}, \bar{a}\}$. There exist infinitely many different $\bar{y}\bar{a} \oplus \bar{y}$.[66]

Now write $\mathfrak{H}_I$, $\mathbf{M}_I$ for its $\mathfrak{H}$, $\mathbf{M}$. We have

$$\overline{\mathbf{M}}_I = \mathbf{R}(U_{\{\bar{x},\bar{a}\}}\ ;\ \bar{x} \in \mathfrak{S},\ \bar{a} \in \mathfrak{G}) \tag{5.5.θ}$$

and by (5.3.β)

$$U_{\{\bar{x},\bar{a}\}}\varphi_{\{\bar{y},\bar{b}\}} = \varphi_{\{\bar{y},\bar{b}\}\{\bar{x},\bar{a}\}^{-1}}. \tag{5.5.ι}$$

We can map $\mathfrak{H}_{\mathfrak{G}S}$ isomorphically on $\mathfrak{H}_I$ by carrying the complete orthonormalized system $F_{\{y,\bar{b}\}}$ of the former into the set of $\varphi_{\{\bar{y},\bar{b}\}}$ of the latter. Then comparison of (5.5.ι) and (5.5.η) shows that this isomorphism carries $\bar{U}_{\bar{a}}\overline{L}_{\omega_{\mathfrak{F}}(x)}$ into $U_{\{\bar{x},\bar{a}\}}$. Hence (5.5.γ) and (5.5.θ) show that it carries $\mathbf{M}$ into $\mathbf{M}_I$. Thus $\mathbf{M}$ and $\mathbf{M}_I$ are spatially isomorphic.

Summing up:

LEMMA 5.5.1. *Given any countably infinite group $\mathfrak{G}$ define S, $\mathfrak{S}$ as described in* (I) *above, and the group $\mathfrak{G}_I = \{\mathfrak{S}, \mathfrak{G}\}$ as described by* (5.5.ζ) (IV) *above. Then the construction of* [5], *pp.* 192–209, *Part IV, can be applied to $\mathfrak{G}$ and S and our construction of* §5.3 *to $\{\mathfrak{S}, \mathfrak{G}\}$ (in place of its $\mathfrak{G}$).*

These two constructions produce spatially isomorphic factors $\mathbf{M}$.

The mappings

$$x\{\bar{y}, \bar{b}\} = x\bar{b} + \bar{y} \tag{5.5.κ}$$

provide a convenient representation of the group $\{\mathfrak{S}, \mathfrak{G}\}$ of (5.5.ζ) in (V). If we specify $x \in S$ then (5.5.κ) appears as a permutation of S. But as it maps $\mathfrak{S}$ on itself, we can also prescribe $x \in \mathfrak{S}$ and view (5.5.κ) as a permutation of $\mathfrak{S}$.

(5.5.κ) makes it clear that $\{\mathfrak{S}, \mathfrak{G}\}$ is the simplest combination of $\mathfrak{S}$ and $\mathfrak{G}$ apart from the "direct group product".

Our above result can also be interpreted in this way:

We start with an arbitrary (countably infinite) group $\mathfrak{G}$ and should like to apply the construction of §5.3. Since $\mathfrak{G}$ may not fulfill the condition (i) of

[66] Put $\bar{y}_b = [\delta_{a,b}\ ,\ a \in \mathfrak{G}],\ \in \mathfrak{S}$. As $\bar{a} \neq 1$, $\bar{y}_b\bar{a} \oplus \bar{y}_b = \bar{y}_c\bar{a} \oplus \bar{y}_c$ means either $b = c$ or $\bar{a}b = c$, $\bar{a}c = b$. In other words: If b runs over all $\mathfrak{G}$ there can never coincide more than two $\bar{y}_b a \oplus \bar{y}_b$. Now $\mathfrak{G}$ is infinite. Hence there are infinitely many different $\bar{y}_b\bar{a} \oplus \bar{y}_b$.

Lemma 5.3.4 this cannot always be done directly. We therefore propose to expand $\mathfrak{G}$ to a group $\mathfrak{G}_I$ which fulfills that condition. (5.5.ζ) provides a very simple generally valid mechanism to do just that. This has the further interesting consequence that within $\mathbf{M}$, the set of group numbers for any $\mathfrak{G}$, the metric closure of a subalgebra is equivalent to the topological closure for any of the ring topologies. For we extend $\mathbf{M}$ to $\mathbf{M}_I$ as above, and Theorem I (of §1.6 above) shows that our statement is true in the extended space $\mathfrak{H}_I$. Since any subalgebra of $\mathbf{M}$ will be reduced by the original space $\mathfrak{H}$, topological closure in $\mathfrak{H}_I$ will yield topological closure. Similarly the discussion of Chapter I may be carried through on the assumption that $\mathbf{M}$ is a group algebra.

§5.6 The analogue of Lemma 5.2.2 is true for the construction of §5.3 too, and this time it is even easier to prove.

LEMMA 5.6.1. *The ring* $\mathbf{M}$ *of* §5.3 *is approximately finite if* $\mathfrak{G}$ *possesses the property* (*i*) *of Lemma* 5.2.2 (*besides the property* (*i*) *of Lemma* 5.3.4).

PROOF. Form the $\mathfrak{G}_1, \mathfrak{G}_2, \cdots$ of (i) in Lemma 5.2.2. Now

$$\mathbf{M} = \mathbf{R}(U_a ; a \in \mathfrak{G}).$$

Hence

$$\mathbf{M} = \mathbf{R}(\mathbf{N}_n ; n = 1, 2, \cdots)$$

with

$$\mathbf{N}_n = \mathbf{R}(U_a ; a \in \mathfrak{G}_n).$$

Thus $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \cdots$ and $\mathbf{N}_n$ is of finite order because $\mathfrak{G}_n$ is finite. Therefore $\mathbf{M}$ is approximately finite by Def. 4.6.1.[46]

Observe that if $\mathfrak{G}$ fulfills (i) in Lemma 5.2.2 (but not necessarily (i) in Lemma 5.3.4) then the group $\mathfrak{G}_I = \{\mathfrak{S}, \mathfrak{G}\}$ of Lemma 5.5.1 possesses that property too. Put $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2 + \cdots$, $\mathfrak{G}_1 \subseteq \mathfrak{G}_2 \subseteq \cdots$ all $\mathfrak{G}_n$ finite. Let $\mathfrak{S}_n$ be the set of all $x = [\bar{\alpha}_a, a \in \mathfrak{G}]$ with $\bar{\alpha}_a = 0$, when a is not in $\mathfrak{G}_n$. Then $\mathfrak{S}_1 \subseteq \mathfrak{S}_2 \subseteq \cdots$ and all the $\mathfrak{S}_n$ are finite. So $\{\mathfrak{S}, \mathfrak{G}\} = \{\mathfrak{S}_1, \mathfrak{G}_1\} + \{\mathfrak{S}_2, \mathfrak{G}_2\} + \cdots$, $\{\mathfrak{S}_1, \mathfrak{G}_1\} \subseteq \{\mathfrak{S}_2, \mathfrak{G}_2\} \subseteq \cdots$ and all the $\{\mathfrak{S}_n, \mathfrak{G}_n\}$ are finite.

Thus it is easy to form groups $\mathfrak{G}$ which fulfill both (i) in Lemma 5.2.2 and (i) in Lemma 5.3.4. The following example is more direct.

LEMMA 5.6.2. *The group* $\tilde{\mathfrak{G}}$ *of all those permutations of* $(1, 2, \cdots)$ *which move only a finite number of elements, fulfills both* (*i*) *in Lemma* 5.2.2 *and* (*i*) *in Lemma* 5.3.4.

PROOF. Ad (i) in Lemma 5.3.4.: Given an $a \in \tilde{\mathfrak{G}}$, $a \neq 1$ choose n so that a moves no number other than $1, \cdots, n$. Assume that a does move i (which is therefore $= 1, \cdots, n$). Let c_j be the transposition of i and j where $j = n + 1, n + 2, \cdots$. Then the only $k = n + 1, n + 2, \cdots$ moved by $c_j^{-1} a c_j$ is $k = j$. Thus the $c_j^{-1} a c_j, j = n + 1, n + 2, \cdots$ are pairwise different and so $\mathfrak{L}_a$ is infinite.

Ad (i) in Lemma 5.2.2.: Let $\tilde{\mathfrak{G}}_n$ be the set of those permutations which move no number other than $1, \cdots, n$. Then $\tilde{\mathfrak{G}} = \tilde{\mathfrak{G}}_1 + \tilde{\mathfrak{G}}_2 + \cdots$, $\tilde{\mathfrak{G}}_1 \subseteq \tilde{\mathfrak{G}}_2 \subseteq \cdots$ and each $\tilde{\mathfrak{G}}_n$ is a finite subset of $\mathfrak{G}$.

The analysis of the invariants (1), (2) of §4.8, gives no new information. As to (1) all examples of [5] as well as those of §5.3 have $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}$. Indeed: Neither of these definitions ever mentions any specific non-real (complex) number. Hence $\mathfrak{H}_c$, $\mathbf{M}_c$ coincide with $\mathfrak{H}$, $\mathbf{M}$ so that $\overline{\mathbf{M}}^c = \overline{\mathbf{M}}` = \overline{\mathbf{M}}$ (cf. §2.3).

Thus we possess no examples of a factor $\mathbf{M}$ in the case (II_1) with $\overline{\mathbf{M}}^c \neq \overline{\mathbf{M}}$. Possibly a generalization of the construction of 5.3, with the introduction of complex numerical factors in (i), (ii) of Lemma 5.3.1 (and (5.3.β) after that lemma) might achieve this.[67] But we have not obtained so far any conclusive results in this respect.

As to (2), $\mathfrak{G}(\overline{\mathbf{M}})$ we know nothing beyond Theorem XV.

Chapter VI. Non-Approximately Finite Factors

§6.1 As pointed out at the end of §4.8 and again at the end of §5.6, we have not succeeded so far to establish the not approximately finite character of any factor with the help of the invariants (1), (2) of §4.8. We shall now prove the existence of not approximately finite factors, but it is necessary to introduce a new invariant in order to achieve this aim.

Throughout what follows, $\mathbf{M}$ is again a factor in case (II_1).

Definition 6.1.1. $\mathbf{M}$ *has the property* Γ *if this is true*:

Given any system, $A_1, \cdots, A_m \in \mathbf{M}$ *and any* $\epsilon > 0$ *there exists a unitary* $U = U(A_1, \cdots, A_m, \epsilon) \in \mathbf{M}$ *with*

$$Tr_{\mathbf{M}}(U) = 0$$

and

$$[[U^{-1}A_kU - A_k]] < \epsilon \quad \textit{for} \quad h = 1, \cdots, m.^{68, 69}$$

This property Γ is clearly a purely algebraic property—i.e. one concerning the algebraic type $\overline{\mathbf{M}}$ only.

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[67] Corresponding to the technique of "multipliers" in forming skew fields over a given center in abstract algebra.

[68] The requirement $Tr_{\mathbf{M}}(U) = 0$ is necessary, since otherwise we could choose $U = 1$.

Outright commutativity of U with $A_1, \cdots, A_m$ (instead of the last inequality) would not do for the following reason: If $\mathbf{M} = \mathbf{R}(A_1, \cdots, A_m)$ then this commutativity would mean $U \in \mathbf{M}'$. Hence $U \in \mathbf{M} \cdot \mathbf{M}'$, $U = \alpha \cdot 1$. Since $Tr_{\mathbf{M}}(U) = 0$ this would mean $U = 0$, contradicting the unitarity of U. Hence no $\mathbf{M} = \mathbf{R}(A_1, \cdots, A_m)$ could then have this property.

Now it can be shown that an approximately finite $\mathbf{M} = \mathbf{R}(A_1, A_2)$ for two suitable A_1, A_2. Hence an approximately finite $\mathbf{M}$ would not have this property—but we need Lemma 6.1.2.

[69] This property could be further subdivided, according to the values of m and of expressions like $\epsilon/\mathrm{Max}\,(|||A_1|||, \cdots, |||A_m|||)$. We could also require $A_1, \cdots, A_m$ to be unitary, etc. For our present purpose, however, these refinements are not needed.

Observe first this

LEMMA 6.1.1. *Let a group* $\mathfrak{G}$ *fulfilling* (i) *in Lemma* 5.3.4 *be given, which possesses this property*:

(i) $\begin{cases} \text{Given any system } a^{(1)}, \cdots, a^{(n)} \in \mathfrak{G} \text{ there exists} \\ a\ c_0 \in \mathfrak{G},\ c_0 \neq 1 \text{ which commutes with } a^{(1)}, \cdots, a^{(n)}. \end{cases}$

Then the $\mathbf{M}$ *of* §5.3 *possesses the property* Γ.

PROOF. Let $A_1, \cdots, A_m \in \mathbf{M}$ and $\epsilon > 0$ be given as indicated in Def. 6.1.1. Following (ii) in Lemma 5.3.6, we can choose $a^{(1)}, \cdots, a^{(n)} \in \mathfrak{G}$ and $\zeta_{h,1}, \cdots, \zeta_{h,n}$ (these are the $\eta_{a^{(1)}}, \cdots, \eta_{a^{(n)}}$ mentioned there for $A = A_h$ for $h = 1, \cdots, m$) so that

$$(6.1.\alpha) \qquad [[A_h - \textstyle\sum_{i=1}^{n} \zeta_{h,i}\, U_{a^{(i)}}]] < \epsilon/2 \qquad \text{for } h = 1, \cdots, m.$$

Now apply our assumption (i) to these $a^{(1)}, \cdots, a^{(n)}$ obtaining a $c_0 \in \mathfrak{G}$, $c_0 \neq 1$ which commutes with them.

Then U_{c_0} and $U_{a^{(i)}}$ commute (cf. footnote [48]) and so

$$U_{c_0}^{-1}(A_h - \textstyle\sum_{i=1}^{n} \zeta_{h,i}\, U_{a^{(i)}})U_{c_0} = U_{c_0}^{-1} A U_{c_0} - \sum_{i=1}^{n} \zeta_{h,i}\, U_{a^{(i)}}.$$

Hence (6.1.α) yields

$$(6.1.\beta) \qquad [[U_{c_0}^{-1} A U_{c_0} - \textstyle\sum_{i=1}^{n} \zeta_{h,i}\, U_{a^{(i)}}]] < \epsilon/2$$

and (6.1.α) and (6.1.β) yield together

$$(6.1.\gamma) \qquad [[U_{c_0}^{-1} A_h U_{c_0} - A_h]] < \epsilon.$$

Clearly $U_{c_0} \sim \{\eta'_{ab^{-1}}\}$ where $\eta'_c \begin{cases} = 1 & \text{for } c = c_0 \\ = 0 & \text{otherwise.} \end{cases}$

As $c_0 \neq 1$ this implies $\eta'_1 = 0$ i.e.

$$(6.1.\delta) \qquad Tr_{\mathbf{M}}(U_{c_0}) = 0.$$

Thus (6.1.γ), (6.1.δ) show that $U = U_{c_0}$ meets all our requirements.

We can now prove

LEMMA 6.1.2. *Approximate finiteness implies the Property* Γ.

PROOF.[70] It suffices to find a group $\mathfrak{G}$ which fulfills the condition (i) in Lemma 5.3.4 (so that the $\mathbf{M}$ of §5.3 is a factor, of course, in case (II_1)), (i) in Lemma 5.2.2 (so that $\mathbf{M}$ is approximately finite by Lemma 5.6.1), and (i) in Lemma 6.1.1 (so that $\mathbf{M}$ possesses the property Γ). Then Theorem XIV and our remark after Def. 6.1.1 take care of everything.

Now the group $\mathfrak{G} = \tilde{\mathfrak{G}}$ of Lemma 5.6.2 possesses the first two properties. As to (i) in Lemma 6.1.1: Let $a^{(1)}, \cdots, a^{(n)}$ be the given elements of $\mathfrak{G}$. Choose p so that none of the $a^{(1)}, \cdots, a^{(n)}$ move any element other than $1, \cdots, p$.

[70] There exists also a relatively simple direct proof, based on Def. 4.1.1.

Let c_0 be the transposition of $p+1$ with $p+2$. Then $c_0 \neq 1$ and it commutes with $a^{(1)}, \cdots, a^{(n)}$.

§6.2 We now proceed to establish the decisive negative result.

LEMMA 6.2.1. *Let a group $\mathfrak{G}$ fulfilling* (i) *in Lemma* 5.3.4 *be given, which possesses this property:*[71]

(i) *There exists a set $\mathfrak{F}_1 \subseteq \mathfrak{G}$ with these properties:*

(i_1) *There exists a $c_1 \in \mathfrak{G}$ such that*

$$\mathfrak{F} + c_1\mathfrak{F}c_1^{-1} = \mathfrak{G} - (1).$$

(i_2) *There exists a $c_2 \in \mathfrak{G}$, such that the three sets $c_2^l\mathfrak{F}c_2^{-l}$, $l = 0, \pm 1$ are disjoint.*

Then the **M** *of* §5.3 *does not possess the property* Γ.

PROOF. Assume the opposite, i.e. that **M** possesses the property Γ. Apply Def. 6.1.1 with $n = 2$. Put $A_1 = U_{c_1}$, $A_2 = U_{c_2}$, while $\epsilon > 0$ will be chosen subsequently. Form the $U = U(A_1, A_2; \epsilon)$ described there.

Then we have:

$$Tr_{\mathbf{M}}(U) = 0 \tag{6.2.α}$$

$$[[U^{-1}U_{c_h}U - U_{c_h}]] < \epsilon \qquad \text{for } h = 1, 2. \tag{6.2.β}$$

As U_{c_h}, U are both unitary and

$$U_{c_h}^{-1}U(U^{-1}U_{c_h}U - U_{c_h}) = U - U_{c_h}^{-1}UU_{c_h}.$$

(6.2.β) is equivalent to

$$[[U - U_{c_h}^{-1}UU_{c_h}]] < \epsilon \qquad \text{for } h = 1, 2. \tag{6.2.γ}$$

Now determine the η_c in the sense of Lemma 5.3.2 for U, $U_{c_h}^{-1}UU_{c_h}$, $U - U_{c_h}^{-1}UU_{c_h}$ in succession. If the first is θ_c it is easy to verify that the second is $\theta_{c_hcc_h^{-1}}$[50] and hence the third is $\theta_c - \theta_{c_hcc_h^{-1}}$. Therefore the application of (i) in Lemma 5.3.6 to U and to $U - U_{c_h}^{-1}UU_{c_h}$ gives

$$[[U]]^2 = \sum_{c\in\mathfrak{G}} |\theta_c|^2$$

$$[[U - U_{c_h}^{-1}UU_{c_h}]]^2 = \sum_{c\in\mathfrak{G}} |\theta_c - \theta_{c_hcc_h^{-1}}|^2.$$

As U is unitary, $[[U]]^2 = 1$. Considering this and (6.2.γ) these equations yield

$$\sum_{c\in\mathfrak{G}} |\theta_c|^2 = 1 \tag{6.2.δ}$$

$$\left(\sum_{c\in\mathfrak{G}} |\theta_c - \theta_{c_hcc_h^{-1}}|^2\right)^{\frac{1}{2}} < \epsilon \qquad \text{for } h = 1, 2. \tag{6.2.ϵ}$$

After these preparations we introduce a measure in $\mathfrak{G}$ by defining

$$\nu(\mathfrak{A}) = \sum_{c\in\mathfrak{A}} |\theta_c|^2 \qquad \text{for } \mathfrak{A} \subseteq \mathfrak{G}.$$

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[71] This proof copies to a certain extent Hausdorf's famous 1/2 − 1/3 division of the sphere. In this connection the use of the free group in Lemma 6.2.2 should be noted.

Then (6.2.δ) becomes

(6.2.ζ) $$\nu(\mathfrak{G}) = 1$$

(6.2.α) means $\theta_1 = 0$ i.e.

(6.2.η) $$\nu((1)) = 0.$$

The triangle inequality in infinitely many dimensions gives

$$|\,(\textstyle\sum_{c\in\mathfrak{A}} |\,\theta_c\,|^2)^{\frac{1}{2}} - (\sum_{c\in\mathfrak{A}} |\,\theta_{c_h c c_h^{-1}}\,|^2)^{\frac{1}{2}}\,| \leqq (\sum_{c\in\mathfrak{G}} |\,\theta_c - \theta_{c_h c c_h^{-1}}\,|^2)^{\frac{1}{2}}.$$

The left-hand side is clearly $|\,\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}\,|$. The right-hand side is $\leqq (\sum_{c\in\mathfrak{G}} |\,\theta_c - \theta_{c_h c c_h^{-1}}\,|^2)^{\frac{1}{2}}$ which is $<\epsilon$ by (6.2.ϵ). So we have

(6.2.θ) $$|\,\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}\,| < \epsilon.$$

Now by (6.2.ζ), $\nu(\mathfrak{A})$ and $\nu(c_h\mathfrak{A}c_h^{-1})$ are $\leqq \nu(\mathfrak{G}) = 1$. Hence

$$\begin{aligned} |\,\nu(\mathfrak{A}) - \nu(c_h\mathfrak{A}c_h^{-1})\,| &= |\,\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}\,|\,(\nu(\mathfrak{A})^{\frac{1}{2}} + \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}) \\ &\leqq 2\,|\,\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}\,| \end{aligned}$$

Therefore (6.2.θ) becomes

(6.2.ι) $$|\,\nu(\mathfrak{A}) - \nu(c_h\mathfrak{A}c_h^{-1})\,| < 2\epsilon.$$

Let us apply (6.2.ι) to $\mathfrak{F}$, c_1, $\mathfrak{F}$, c_2, $c_2^{-1}\mathfrak{F}c_2$, c_2 in place of its $\mathfrak{A}$, c_h. Then

(6.2.κ) $$|\,\nu(\mathfrak{F}) - \nu(c_1\mathfrak{F}c_1^{-1})\,| < 2\epsilon$$

(6.2.λ) $$|\,\nu(\mathfrak{F}) - \nu(c_2\mathfrak{F}c_2^{-1})\,| < 2\epsilon$$

(6.2.μ) $$|\,\nu(c_2^{-1}\mathfrak{F}c_2) - \nu(\mathfrak{F})\,| < 2\epsilon$$

obtain. Now (i_1) and (6.2.ζ), (6.2.η) and (6.2.κ) give

$$\nu(\mathfrak{F}) + (\nu(\mathfrak{F}) + 2\epsilon) > 1$$

i.e.

(6.2.ν) $$\nu(\mathfrak{F}) > \tfrac{1}{2} - \epsilon.$$

On the other hand (i_2) and (6.2.ζ), (6.2.λ) and (6.2.μ) give

$$\nu(\mathfrak{F}) + (\nu(\mathfrak{F}) - 2\epsilon) + (\nu(\mathfrak{F}) - 2\epsilon) < 1$$

i.e.

(6.2.o) $$\nu(\mathfrak{F}) < \tfrac{1}{3} + \tfrac{4}{3}\epsilon.$$

(6.2.ν) and (6.2.o) imply

$$\tfrac{1}{2} - \epsilon < \tfrac{1}{3} + \tfrac{4}{3}\epsilon \quad \text{or} \quad 1/14 < \epsilon.$$

Hence it suffices to choose $\epsilon = 1/14$ in order to have a contradiction.

Thus we have shown that **M** cannot possess the property Γ.

COROLLARY 1. *Under the above assumptions,* **M** *is not approximately finite.*

PROOF. Combine the above lemma with Lemma 6.1.2.

COROLLARY 2. *We may replace condition* (i_2) *on* $\mathfrak{G}$ *and* $\mathfrak{F}$ *by*

(i_2'). *There exist two elements* c_2 *and* c_3 *in* $\mathfrak{G}$ *such that* $\mathfrak{F}$, $c_2\mathfrak{F}c_2^{-1}$, $c_3\mathfrak{F}c_3^{-1}$ *are mutually exclusive.*

PROOF. In the first paragraph of the above proof we let $A_3 = U_{c_3}$ and then take $U = U(A_1, A_2, A_3, \epsilon)$ instead of $U = U(A_1, A_2, \epsilon)$. We have $h = 1, 2, 3$ in (6.2.β), (6.2.γ), (6.2.ϵ), (6.2.θ) and (6.2.ι), instead of $h = 1, 2$. Finally we obtain an equation

$$(6.2.\mu') \qquad |\nu(\mathfrak{F}) - \nu(c_3\mathfrak{F}c_3^{-1})| < 2\epsilon$$

instead of (6.2.μ), by applying (6.2.ι) to $\mathfrak{F}$, c_3 instead of $c_2^{-1}\mathfrak{F}c_2$, c_2. (6.2.μ') is used instead of (6.2.μ) in order to obtain (6.2.o), and the rest of the proof is the same.

We conclude by giving a specific instance of an **M** which does not possess the property Γ and is not approximately finite. We do this by giving an example of a group $\mathfrak{G}$ which fulfills the requirements of Lemma 6.2.1.

LEMMA 6.2.2. *Let* $\mathfrak{G}$ *be the free group of two generators* c_1, c_2. *Then* $\mathfrak{G}$ *fulfills the requirements of Lemma* 6.2.1.

PROOF. Ad (i) in Lemma 5.3.4: Consider an $a \in \mathfrak{G}$, $a \neq 1$. Write a as a power product of c_1, c_2 of minimum length. Then it is easy to verify that $c_1^{-n}ac_1^n$, $n = 1, 2, \cdots$ are pairwise different unless $a = c_1^m$. In the latter case we have that the $c_2^{-n}ac_2^n = c_2^{-n}c_1^m c_2^n$ are distinct unless $m = 0$ i.e. $a = 1$. Thus unless $a = 1$, $\mathfrak{L}_a$ is infinite.

Ad (i) in Lemma 6.2.1: Let $\mathfrak{F}$ be the set of $a \in \mathfrak{G}$ which when written as a power product of c_1, c_2 of minimum length end with a c_1^n, $n = \pm 1, \pm 2, \cdots$. Then (i_1) and (i_2) in Lemma 6.2.1, i.e. (i) eod., are immediately verified.

Combining Lemma 6.2.2 with Corollary 1 to Lemma 6.2.1 gives

THEOREM XVI. *There exist not approximately finite factors* **M** *in the case* (II_1).

Or, considering Theorem XV,

THEOREM XVI'. *There exist in case* (II_1) *more than one algebraical type* $\overline{\mathbf{M}}$.

§6.3 We wish in this section to present a number of examples of groups $\mathfrak{G}$ for which the corresponding **M** of §5.3 is not approximately finite.

LEMMA 6.3.1. *Let* $\mathfrak{G}$ *be a group which contains two multiplicative subsets* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$. *Suppose that* $\mathfrak{G}$ *is the free product of* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *and that* $\mathfrak{F}_1$ *contains at least two elements and* $\mathfrak{F}_2$ *contains at least three. Then* $\mathfrak{G}$ *satisfies condition* (i) *of Lemma* 5.3.4, *condition* (i_1) *of Lemma* 6.2.1 *and condition* (i_2') *of Corollary* 2 *of that lemma (for the same* $\mathfrak{F}$).

PROOF. Let c_1 be an element of $\mathfrak{F}_1$ which is not 1, and c_2, c_3 elements of $\mathfrak{F}_2$ which are each not 1.

We prove first that (i) of Lemma 5.3.4 holds, i.e. that for $a \in \mathfrak{G}$, $a \neq 1$ the $\mathfrak{L}_a$ is infinite.

Suppose an $a \in \mathfrak{G}$, $a \neq 1$ is given. Since $\mathfrak{G}$ is a free product of $\mathfrak{F}_1$ and $\mathfrak{F}_2$,

each a has a unique factorization $a_1 \cdot a_2 \cdot \cdots \cdot a_p$ into a product in which the factors are alternately in $\mathfrak{F}_1$ and $\mathfrak{F}_2$ and no factor is 1.

We note for later use that if $c \in \mathfrak{F}_i$, then $c^{-1} \in \mathfrak{F}_i$. Suppose $i = 1$. Then c^{-1} will have a factorization $b_1 \cdot b_2 \cdot \cdots \cdot b_r$. If $b_1 \in \mathfrak{F}_2$ we have $1 = cc^{-1} = c \cdot b_1 \cdot b_2 \cdot \cdots \cdot b_r \neq 1$. Thus $b_1 \in \mathfrak{F}_1$. But $1 = cc^{-1} = (cb_1) \cdot b_2 \cdot \cdots \cdot b_r$ implies $r = 1$. Consequently $c^{-1} = b_1 \in \mathfrak{F}_1$.

Now suppose firstly that a_1 (the first factor) and a_p (the last factor) are both from $\mathfrak{F}_1$. Let $b_n = (c_2c_1)^n$. Then the set of $b_n^{-1}ab_n$ are for $n = 1, 2, \cdots$ all distinct. (The factorization is immediately apparent.) Hence $\mathfrak{L}_a$ is infinite in this case.

If a_1 and a_p are both from $\mathfrak{F}_2$, we let $b_n = (c_1c_2)^n$ and the result is again apparent.

If a_1 is from $\mathfrak{F}_1$ and a_p is from $\mathfrak{F}_2$, we consider c_2 and c_3 and denote by c' that one of c_2, c_3 which is not equal to a_p^{-1}. Let $b_n = (c'c_1)^n$. Then for $n = 1, 2, \cdots$ the factorization of $b_n^{-1}ab_n$ is obtained by simply writing out the expressions for these elements and coalescing a_p and the first c' in $b_n.a_pc'$ is in $\mathfrak{F}_2$ but is not 1 since $c' \neq a_p^{-1}$. Hence the factorization has been obtained. These factorizations for $n = 1, 2, \cdots$ show that the $b_n^{-1}ab_n$ are distinct, and hence $\mathfrak{L}_a$ is infinite.

If a_1 is in $\mathfrak{F}_2$ and a_p is in $\mathfrak{F}_1$ we define c' as that one of c_2, c_3 which is not equal to a_1. An argument similar to that of the preceding paragraph will show that in this case also $\mathfrak{L}_a$ is infinite.

Thus we have established (1) of Lemma 5.3.4.

To establish (i_1) of Lemma 6.2.1 and (i_2') of Corollary 2 of that lemma for the same $\mathfrak{F}$, we define $\mathfrak{F}$ as the set of a's, $a \neq 1$ having a factorization $a_1 \cdot \cdots \cdot a_p$ in which a_p is in $\mathfrak{F}_1$.

We show firstly (i_1) in Lemma 6.2.1, i.e.

$$\mathfrak{F} + c_1^{-1}\mathfrak{F}c_1 = \mathfrak{G} - (1).$$

If $a \in \mathfrak{G}$, $a \neq 1$ then a has a factorization $a_1 \cdot \cdots \cdot a_p$. Now if a is not $\in \mathfrak{F}$, a_p is not $\in \mathfrak{F}_1$ but $a_p \in \mathfrak{F}_2$. In this case we consider $c_1ac_1^{-1} = c_1a_1 \cdot \cdots \cdot a_pc_1^{-1}$. If p is even, i.e. $p = 2, 4, \cdots$ then $a_1 \in \mathfrak{F}_1$ (since $a_p \in \mathfrak{F}_2$) and the factorization of $c_1ac_1^{-1}$ is either $(c_1a_1) \cdot a_2 \cdot \cdots \cdot a_p \cdot c_1^{-1}$ or (if $c_1a_1 = 1$) $a_2 \cdot \cdots \cdot a_pc_1^{-1}$. If p is odd, i.e., $p = 1, 3, \cdots$, $a_1 \in \mathfrak{F}_2$ and $c_1ac_1^{-1}$ has the factorization $c_1 \cdot a_1 \cdot \cdots \cdot a_p \cdot c_1^{-1}$. These factorizations all show that $c_1ac_1^{-1}$ is $\in \mathfrak{F}$. Thus if $a \neq 1$ and a is not $\in \mathfrak{F}$, $c_1ac_1^{-1} \in \mathfrak{F}$ or $a \in c_1^{-1}\mathfrak{F}c$. This is equivalent to $\mathfrak{G} - (1) \subseteq \mathfrak{F} + c_1^{-1}\mathfrak{F}c_1$. However this inclusion is not proper. For 1 is not in $\mathfrak{F}$. Consequently 1 is not in $c_1^{-1}\mathfrak{F}c_1$ and $\mathfrak{F} + c_1^{-1}\mathfrak{F}c_1$. We can now conclude

$$\mathfrak{F} + c_1^{-1}\mathfrak{F}c_1 = \mathfrak{G} - (1).$$

One can establish (i_2') of Corollary 2 of Lemma 6.2.1 in a similar (indeed easier) way.

Thus our Lemma has been proven.

This lemma permits us to offer many other examples similar to the $\mathfrak{G}$ of Lemma

6.2.2. For we may obtain $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in a number of ways, and then $\mathfrak{G} = \mathfrak{G}(\mathfrak{F}_1, \mathfrak{F}_2)$ is constructed by adding to 1 the "free" products $a_1 \cdot a_2 \cdot \cdots \cdot a_p$ where the a_i's are never 1 and are alternately from $\mathfrak{F}_1$ and $\mathfrak{F}_2$. As an example we may take $\mathfrak{F}_1$ as the group generated by a single generator a of order $p = 2, 3, \cdots, \infty$ and $\mathfrak{F}_2$ as the group generated by an element b of order $q = 3, 4, \cdots, \infty$. $\mathfrak{G}(a, b) = \mathfrak{G}(\mathfrak{F}_1, \mathfrak{F}_2)$ is then an example. Of course we may take for $\mathfrak{F}_1$ any group of order $\geqq 2$ and $\mathfrak{F}_2$ of order $\geqq 3$.

The restriction that $\mathfrak{F}_2$ contains at least 3 members rules out the case in which $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are both groups of order 2, i.e. $\mathfrak{F}_1$ consists of $(1, a)$ and $\mathfrak{F}_2$ consists of $(1, b)$. Here of course we do not have the (i_2) of Lemma 6.2.1 or the (i_2') of its second corollary. However, condition (i) of Lemma 5.3.4 also fails. For the element ab has only ba and itself as conjugates, i.e. $\mathfrak{L}_{ab} = (ab, ba)$. Thus $\mathfrak{L}_{ab}$ is finite. Hence by Lemma 5.3.4, the **M** for this $\mathfrak{G}$ is not even a factor.

Another lemma, which we shall use in an appendix, is readily shown now.

LEMMA 6.3.2. *If $\mathfrak{G}$ is such that to every $a \in \mathfrak{G}$, $a \neq 1$ there is a $b \in \mathfrak{G}$, b of order $\geqq 3$ such that $\mathfrak{G}$ contains the free group $\mathfrak{G}(a, b)$ then $\mathfrak{G}$ fulfills* (i) *of Lemma* 5.3.4. (*If a is not of order 2, b need only differ from the identity.*)

PROOF. Let $\bar{\mathfrak{L}}_a$ denote the class of a relative to the subgroup $\mathfrak{G}(a, b)$. Our discussion following Lemma 6.3.1 shows that $\mathfrak{G}(a, b)$ satisfies the hypotheses of Lemma 6.3.1. Consequently that Lemma and Lemma 5.3.4 imply that $\bar{\mathfrak{L}}_a$ is infinite. Since $\bar{\mathfrak{L}}_a \subseteq \mathfrak{L}_a$ we have $\mathfrak{L}_a$ infinite for $\mathfrak{G}$ itself, and thus (i) of Lemma 5.3.4 holds.

APPENDIX[72]

In this appendix we give examples of two Class (II_1) factors which are not isomorphic and yet each of which is isomorphic to a subset of the other.

The factors will be the factors associated with certain groups $\mathfrak{G}_1$ and $\mathfrak{H}_1$ as in §5.3. These groups are best presented as subgroups of a group $\mathfrak{G}$ which we shall now define.

$\mathfrak{G}$ is determined by generators x_r one for each rational number $r = p/q$. The commutation rules for these generators are determined as follows. Let A denote the set of dyadic rationals $q/2^n$ in the most reduced form where $n = 0, 1, 2, \cdots$ and q is odd except possibly when $n = 0$. Let B denote the set of all other rationals.

The commutation rules for x_r are the following:

(i) If $r \in A$ then x_r combines freely with any product $x_{s_1} \cdot x_{s_2} \cdot \cdots \cdot x_{s_p}$ if $s_1, s_2, \cdots, s_p$ are all $< r$.

(ii) If $r \in B$, x_r commutes with all products $x_{s_1} \cdot x_{s_2} \cdot \cdots \cdot x_{s_p}$ if $s_1, s_2, \cdots, s_p$ are all $< r$.

(iii) All x_r are of order ∞.

One can show that the set of products $x_{r_1}^{n_1} \cdot \cdots \cdot x_{r_p}^{n_p}$, $n_i = \pm 1, \pm 2, \cdots$ (with

[72] This example was originally part of a later investigation by the second-named author, but it can be most easily presented here, using our present methods.

a finite number of factors) constitutes a denumerably infinite group $\mathfrak{G}$.[73] This is also true for the set of products in which the x_r's are restricted, so that r is in a fixed non-empty subset S of the rational numbers.

Let $\mathfrak{H}_r$ be the subgroup of $\mathfrak{G}$ determined by the generators $x_{r'}$ with $r' \leqq r$. Let $\mathfrak{G}_r$ be the subgroup of $\mathfrak{G}$ determined by the generators $x_{r'}$ with $r' < r$. We can show

LEMMA A.1. (i) $\mathfrak{G}_r \subseteq \mathfrak{H}_r$

(ii) *If* $r' < r$, $\mathfrak{H}_{r'} \subseteq \mathfrak{G}_r$.

(iii) *If* $n = \pm 1, \pm 2, \cdots$ *then* $\mathfrak{H}_{r+n}$ *is isomorphic to* $\mathfrak{H}_r$.

(iv) $\mathfrak{G}_r$ *has property* (i) *of Lemma* 5.3.4.

(v) $\mathfrak{G}_r$ *has property* (i) *of Lemma* 6.1.1.

(vi) *Assume* $r \epsilon A$. *Then* $\mathfrak{H}_r$ *satisfies the hypotheses of Lemma* 6.3.1, *and consequently it satisfies* (i_1) *and* (i_2') *of Lemma* 6.2.1 *and its Corollary* 2 *and* (i) *in Lemma* 5.3.4.

PROOF. Ad (i) and (ii) are clear.

Ad (iii). The correspondence $x_{r+n} \sim x_r$ between the generators of $\mathfrak{H}_{n+r}$ and $\mathfrak{H}_r$ is one-to-one and preserves the properties $r \epsilon A$ and $r \epsilon B$. Consequently it determines an isomorphism between $\mathfrak{H}_{n+r}$ and $\mathfrak{H}_r$.

Ad (iv). Suppose $a \epsilon \mathfrak{G}_r$. Then $a = x_1^{n_1} \cdot \cdots \cdot x_p^{n_p}$ with each $r_i < r$. Let r' be the $\mathrm{Max}_{i=1,\cdots,p} r_i$. Then of course $r' < r$ and there is an $r'' \epsilon A$ such that $r' < r'' < r$. By the first commutation rule above $x_{r''}$ combines freely with a. By the definition of $\mathfrak{G}_r$, $\mathfrak{G}(a, x_{r''}) \subseteq \mathfrak{G}_r$. The third rule above tells us that x_r is of infinite order. These last two results show that the hypotheses of Lemma 6.3.2 are satisfied. It follows from that lemma that the condition (i) of Lemma 5.3.4 is satisfied.

Ad (v). $\mathfrak{G}_r$ has property (i) of Lemma 6.1.1: For let $a_1, \cdots, a_k$ be any k elements of $\mathfrak{G}_r$ and let $x_{r_1}, x_{r_2}, \cdots, x_{r_l}$ denote the generators which appear explicitly in $a_1, \cdots, a_k$. Let $r' = \mathrm{Max}_{i=1,\cdots,l} r_i$. Then $r' < r$ by the definition of $\mathfrak{G}_r$. Hence there is an $r'' \epsilon B$ such that $r' < r'' < r$. Let $c = x_{r''}$. Then $c \neq 1$, $c \epsilon \mathfrak{G}_r$ and c commutes with $a_1, \cdots, a_k$ by the second commutation rule above. This result shows that (i) in Lemma 6.1.1 is satisfied.

Ad (vi). For $\mathfrak{H}_r$ we let $\mathfrak{F}_1 = \mathfrak{G}_r$ and $\mathfrak{F}_2 = (x_r^n)$. Now x_r is of infinite order and the first commutation rule shows that $\mathfrak{H}_r$ is the free product of $\mathfrak{G}_r$ and $\mathfrak{F}_2$. These show that the hypotheses of Lemma 6.3.1 are fulfilled. That lemma itself

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[73] This can be most readily shown by obtaining a "normal form" for each such product $x_{r_1}^{n_1} \cdot \cdots \cdot x_{r_p}^{n_p}$. This normal form may be obtained as follows. Let r_i denote the largest subscript which appears and is in B. We move $x_{r_i}^{n_i}$ as far to the left as possible. Let r_i' be the next largest subscript, which is also in B. We then move $x_{r_i'}^{n_i'}$ as far to the left as possible. We continue on until we have considered all the $r_i \epsilon B$. The resulting product is then in a form in which the $x_{r_j}^{n_j}$, $r_j \epsilon A$ have the same order as before, but each such $x_{r_j}^{n_j}$, $r_j \epsilon A$ is followed by a product $x_{r_{i_1}}^{n_{i_1}} \cdot \cdots \cdot x_{r_{i_k}}^{n_{i_k}}$ with the r_{i_t} each in B, each $< r_j$ and with $r_{i_1} < r_{i_2} < \cdots < r_{i_k}$.

We may use this form to establish the associativity of the composition rule. The existence of an inverse is clear.

now shows that (i_1) and (i_2') of Lemma 6.2.1 and its Corollary 2 are satisfied, and (i) in Lemma 5.3.4.

We next state a quite general lemma.

LEMMA A.2. *Let* $\mathbf{M}$ *be a ring which contains* $(\alpha 1)$ *and suppose* $f \in \mathfrak{H}$ *is such that* $Af = 0$ *and* $A \in \mathbf{M}$ *imply* $A = 0$. *Let* $\mathfrak{M} = \mathfrak{M}_f^{\mathbf{M}}$. *Then if* A *is closed, has domain dense, and* η $\mathbf{M}$, A *is bounded if and only if it is bounded on* $\mathfrak{M}$.

PROOF. It is clear that we need only show that if A is unbounded, it is unbounded on $\mathfrak{M}$. Let $A = WB$ where W is partially isometric and B is definite (cf. [5], §4.4). Since $A \; \eta \; \mathbf{M}$, we have $B \; \eta \; \mathbf{M}$ and also for all g, $|Ag| = |Bg|$. (This may be ∞ of course.) Hence we may substitute B for A, or what is the same thing assume that A is definite. But A definite implies that there is a resolution $\{E(\lambda)\} \subset \mathbf{M}$ such that $A = \int_0^\infty \lambda \, dE(\lambda)$. Since A is unbounded, $1 - E(\lambda) \neq 0$ for $\lambda < \infty$. Let $g = (1 - E(\lambda))f$ for any fixed λ. By our hypothesis on f, $g \neq 0$ since $1 - E(\lambda) \neq 0$. From the formula for A we see $|Ag| > \lambda |g|$. Furthermore $g \in \mathfrak{M}_f^{\mathbf{M}} = \mathfrak{M}$. Thus if A is unbounded, it is unbounded on $\mathfrak{M}$.

COROLLARY. *If* $A \in \mathbf{M}$ *the bound of* A *on* $\mathfrak{M}$ *is the same as the bound of* A ($|||A|||$).

PROOF. It is clear that the bound of A on $\mathfrak{M}$ is $\leqq |||A|||$. To show that it can't be less one uses an argument similar to the above.

We wish to apply Lemma A.2 to the case in which $\mathbf{M}$ is a subring of the set of group numbers in the sense of §5.3. We may do this by taking $f = \varphi_1$ in the sense of (5.3.α). For if $A \in \mathbf{M}$, we have that $A \sim \{\eta_{a^{-1}b}\}$ by Lemma 5.3.2 and $A\varphi_1 = \sum_{c \in \mathfrak{G}} \eta_c \varphi_c$ by $({}^{49}, \gamma)$. Thus $A\varphi_1 = 0$ implies $\eta_c = 0$ for all $c \in \mathfrak{G}$ and hence $A = 0$.

It is also useful to notice in applying this to group numbers that if we have any sequence $\{\eta_c\}$ with $\sum_{c \in \mathfrak{G}} |\eta_c|^2 < \infty$ then the matrix $\{\eta_{a^{-1}b}\}$ determines an A with a closure and domain dense. For it is clear from $({}^{49}, \gamma)$ that A is defined for all φ_a, $a \in \mathfrak{G}$ of (5.3.α) and hence for the linear combinations of these. Since the latter are dense, A has domain dense. Furthermore the matrix $\{\bar{\eta}_{b^{-1}a}\}$ is adjoint to it. Since this latter also has domain dense, A must have a closure. We shall consider A to be the least closed transformation, with matrix $\{\eta_{a^{-1}b}\}$. On its domain, A is the limit of finite linear combinations of the U_c and hence is η $\mathbf{M}$. Cf. Lemmas 5.3.1, 5.3.2 and 5.3.6.

LEMMA A.3. *Suppose* $\mathfrak{G}^0$ *and* $\mathfrak{G}^{00}$ *are two groups and* $\mathbf{M}^0$ *and* $\mathbf{M}^{00}$ *their respective group numbers in the sense of* §5.3. *Then if* $\mathfrak{G}^0 \subseteqq \mathfrak{G}^{00}$, $\mathbf{M}^0$ *is algebraically isomorphic with a subring of* $\mathbf{M}^{00}$.

PROOF. For an $A \in \mathbf{M}^{00}$ we have a sequence $\{\eta_c ; c \in \mathfrak{G}^{00}\}$ as in Lemma 5.3.2. Let $\mathbf{M}^{0+}$ denote the set of A's $\in \mathbf{M}^{0,0}$ for which $\eta_c = 0$ if c not $\in \mathfrak{G}^0$. Since $\mathfrak{G}^0$ is a subgroup, this set of A's is closed under the operations αA, A^*, $A + B$, AB. Furthermore this set of A's is metrically closed, and as we mentioned at the end of §5.5, for a subalgebra of group numbers, this implies ring closure. Thus $\mathbf{M}^{0+}$ is a ring, and indeed is the ring determined by the U_c, $c \in \mathfrak{G}^0$.

Let $\mathfrak{M}^0$ be the set $\subseteq \mathfrak{H}^{0,0}$ determined by the φ_a with $a \in \mathfrak{G}^0$ (cf. 5.3.α). It is clear that we can identify this with $\mathfrak{H}^0$ the space on which we have $\mathbf{M}^0$ defined. Now for every $A \in \mathbf{M}^0$ we have a sequence $\{\zeta_c\,;\, c \in \mathfrak{G}^0\}$ and corresponding matrix $\{\zeta_{a^{-1}b}\}$. Correspondingly, we have a sequence $\{\eta_c\,,\ c \in \mathfrak{G}^{00}\}$ with $\eta_c = \zeta_c$ for $c \in \mathfrak{G}^0$ and $\eta_c = 0$ otherwise. As we pointed out above, the matrix $\{\eta_{a^{-1}b}\}$ determines a closed operator $\tilde{A}$ with domain dense. This is clearly $\eta\,\mathbf{M}^{0+}$ and hence by the application of Lemma A.2 described above, it is bounded if and only if it is bounded on $\mathfrak{M}_f^{\mathbf{M}^{0+}}$. Now one can show by our above definition of $\mathbf{M}^{0+}$ that $\mathfrak{M}_f^{\mathbf{M}^{0+}} = \mathfrak{M}^0$. Now $\tilde{A} = A$ on $\mathfrak{M}^0$ and thus is bounded there. Hence, by Lemma A.2, $\tilde{A}$ is bounded and $\in \mathbf{M}^{0+}$. We now see that to every $A \in \mathbf{M}^0$ there corresponds an $\tilde{A} \in \mathbf{M}^{0+}$ such that $\tilde{A}$ agrees with A on $\mathfrak{H}^0 = \mathfrak{M}^0$. On the other hand, it is clear that to every $\tilde{A} \in \mathbf{M}^{0+}$ we have an $A \in \mathbf{M}^0$ with this property.

One can readily verify that this is an algebraic isomorphism.

We are now ready for the final step. Let $\mathbf{M}_1$ be the set of group numbers for $\mathfrak{G}_1$, $\mathbf{M}_2$ the set of group numbers for $\mathfrak{H}_1$ (cf. §5.3).

LEMMA A.4. (i) $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are factors in Case* (II$_1$).

(ii) $\mathbf{M}_1$ *is algebraically isomorphic with a subring of* $\mathbf{M}_2$.

(iii) $\mathbf{M}_2$ *is algebraically isomorphic with a subring of* $\mathbf{M}_1$.

(iv) $\mathbf{M}_1$ *has property* Γ *of Def.* 6.1.1.

(v) $\mathbf{M}_2$ *does not have property* Γ.

PROOF. Ad (i). $\mathbf{M}_1$ is a factor by Lemma A.1 (iv) and Lemma 5.3.4. $\mathbf{M}_2$ is a factor by Lemma A.1 (vi) and Lemma 5.3.4. $\mathbf{M}_1$ and $\mathbf{M}_2$ are both in Case (II$_1$) by Lemma 5.3.5 (i).

Ad (ii). This follows from Lemma A.1 (i) and Lemma A.3.

Ad (iii). Let $\mathbf{M}_0$ denote the set of group numbers for $\mathfrak{H}_0$. Lemma A.1 (iii) for $n = -1$ shows that $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are isomorphic, and this of course implies that $\mathbf{M}_2$ is algebraically isomorphic to $\mathbf{M}_0$. Lemma A.1 (ii) and Lemma A.3 imply that $\mathbf{M}_0$ is isomorphic to a subring of $\mathbf{M}_1$. Hence $\mathbf{M}_2$ is algebraically isomorphic to a subring of $\mathbf{M}_1$.

Ad (iv). This is implied by Lemma A.1 (v) and Lemma 6.1.1.

Ad (v). This is implied by Lemma A.1 (vi) and Corollary 2 of Lemma 6.2.1.

Since property Γ is an isomorphism invariant, (iv) and (v) imply that $\mathbf{M}_1$ and $\mathbf{M}_2$ are not algebraically isomorphic. Combining this with (ii), (i) and (iii), we have

THEOREM A. *There exist two rings* $\mathbf{M}_1$, $\mathbf{M}_2$, *each a factor in Case* (II$_1$), *which are not algebraically isomorphic to each other but are such that each is algebraically isomorphic to a subring of the other.*

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ERRATUM

Page 318, lines 12 and 13 (after semicolon). Read: "*and therefore, if B is completely additive (i. e., if the union of the elements of every subset of B exists), then it is isomorphic with a direct sum of f-homogeneous Boolean algebras.*"

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TABLE OF CONTENTS

On Hodge's theory of harmonic integrals. By HERMANN WEYL 1
Moutard-Čech hyperquadrics associated with a point of a hypersurface. By BUCHIN-SU 7
The class-ring in multiplicative systems. By A. R. RICHARDSON 21
On homotopy type and deformation retracts. By RALPH H. FOX 40
On the deformation retraction of some function spaces associated with the relative homotopy groups. By RALPH H. FOX 51
On permutation groups of prime degree and related classes of groups. By RICHARD BRAUER 57
On the convergence of continued fractions to meromorphic functions. By WALTER LEIGHTON and W. J. THRON 80
On the continuum hypothesis. By ISAIAH MAXIMOFF 90
Linearly arc-wise connected topological abelian groups. By R. C. JAMES . 93
Contributions to the analytic theory of continued fractions and infinite matrices. By E. D. HELLINGER and H. S. WALL 103
Remarks on two-leaved orientable covering manifolds of closed manifolds. By TSAI-HAN KIANG 128

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1
7
21
40
51
57
80
90
93
103
128

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TABLE OF CONTENTS

On the non-existence of regular stationary solutions of relativistic field equations. By A. EINSTEIN and W. PAULI 131

On the sum of two sets of integers. By E. ARTIN and PETER SCHERK 138

Contributions to the theory of the Dirichlet L-series and the Epstein zeta-functions. By CARL SIEGEL 143

On continuous path surfaces of zero area. By T. RADÓ 173

Automorphism rings of primary abelian operator groups. By REINHOLD BAER 192

Theorems on linear combinatorial topology and general measure. By R. RADO 228

Family of totally umbilical hypersurfaces in an Einstein space. By YUNG-CHOW WONG 271

Euclidean applications of the projective differential geometry of the R_λ-correspondent. By P. O. BELL and W. C. FOREMAN 298

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TABLE OF CONTENTS

The variational theory in the large including the non-regular case—first paper. By GEORGE EWING and MARSTON MORSE 339
The variational theory in the large including the non-regular case—second paper. By GEORGE EWING and MARSTON MORSE 354
Analytic solutions of non-linear difference equations. By WALTER STRODT 375
A note on Cesaro summability of Fourier series. FU TRAING WANG 397
The inverse Nörlund Mean. By FLORENCE M. MEARS 401
Convergence of certain gap series. By M. KAC 411
On the expansion of the partition function in a series. By HANS RADEMACHER 416
Sur la distribution limite du terme maximum d'une série aléatoire. Par B. GNEDENKO 423
Transformation groups of spheres. By DEANE MONTGOMERY and HANS SAMELSON 454
Extensions of topological spaces. By S. FOMIN 471
Foundations of the theory of Lie groups with real parameters. By P. A. SMITH 481
Combinations of closure relations. By OYSTEIN ORE 514
Some remarks on algebras over an algebraically closed field. By C. NESBITT and W. M. SCOTT 534
Open additive semi-groups of complex numbers. By EINAR HILLE and MAX ZORN 554
On a projective invariant of a non-holonomic surface. By HSIEN-CHUNG WANG 562

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339
354
375
397
401
411
416
423
454
471
481
514
534
554
562

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TABLE OF CONTENTS

On the importance of the relation [(A, B), (A, C)] < [(B, C), (C, A), (A, B)] between three elements of a structure. By HENRY LÖWIG 573

On the theory of partially ordered linear systems and linear spaces. By DIMITRY WASSILKOFF 580

Homology with local coefficients. By N. E. STEENROD 610

Non-commutative chains, II. By W. W. FLEXNER 628

Some remarks on set theory. By P. ERDÖS 643

Corrections to two of my papers. (Addendum by EINAR HILLE.) By PAUL ERDÖS 647

Analytic and meromorphic continuation by means of Green's Formula. By S. BOCHNER 652

Discontinuous groups. By CARL LUDWIG SIEGEL 674

On the theory of local rings. By CLAUDE CHEVALLEY 690

On some algebraical properties of operator rings. By JOHN VON NEUMANN .. 709

On rings of operators. IV. By F. J. MURRAY AND J. VON NEUMANN 716

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573

580
610
628
643

647

652
674
690
709
716